NOTES FOR MARCH 29

BROOKE ULLERY

Today, X is compact Kähler:

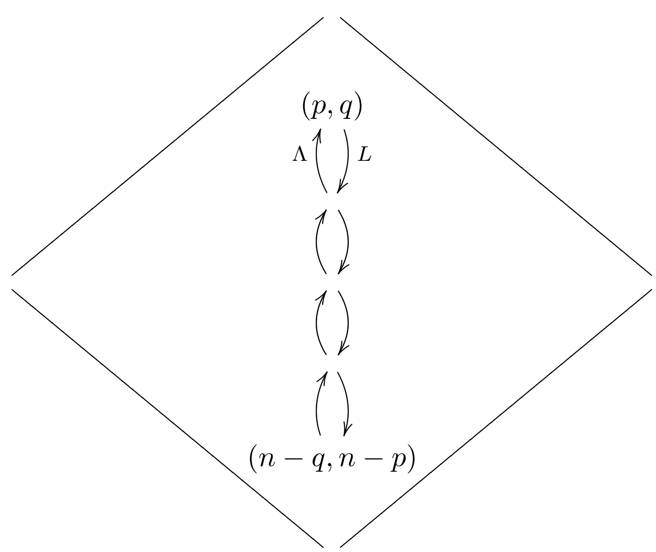
Primitive cohomology and Lefschetz decomposition

Question. *How do L and Λ act on harmonic forms?*

Here is the answer, which we will be proving today:

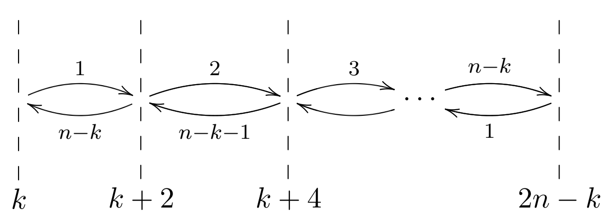
Answer. *L and Λ commute with Δ (Problem 2 on homework). They take harmonic forms to harmonic forms (and, more generally, the λ-eigenspace of Δ to itself).*

The action splits into:



where all arrows are isomorphisms. In correct bases, they are integers times the identity.

If the string goes from H^k to H^{2n-k}, the arrows are



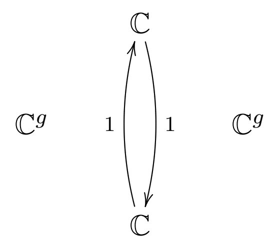
The longest string is the one that goes from H^0, represented by the harmonic function 1, to the volume form $\omega^n/n!$.

This decomposition respects Hodge decomposition, so:

$$\mathcal{H}^{\bullet,\bullet} = \bigoplus_{p,q} (\text{string from } (p,q) \text{ to } (n-q, n-p))$$

Example 1. Consider projective space. The only cohomology groups are in $H^{p,p}$, which is 1 dimensional. So we have a vertical row of $\mathbb{C}$'s with bases 1, ω, $\omega^2/2$, $\omega^3/6$, ...

Example 2. A genus g curve. The Hodge diamond looks like:



Definition 3. A harmonic form is called ***primitive*** if it is in $\ker \Lambda$ (i.e. the "top of a string"). We also call such a cohomology class primitive.

So we have

$$H^{p,q} = H^{p,q}_{\text{prim}} \oplus LH^{p-1,q-1}_{\text{prim}} \oplus L^2 H^{p-2,q-2}_{\text{prim}} \oplus \cdots$$

Theorem 4 (Hard Lefschetz)**.** $L^{n-k} : H^k(X) \to H^{2n-k}(X)$ *is an isomorphism.*

Key Computation: $[\Lambda, L] = (n-k)\text{Id}$ on $\Omega^k(X)$.

Proof. We can immediately reduce to nice coordinates. The formula is C^∞-linear, so it suffices to check for $dz_I \wedge d\bar{z}_J$.

$$dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_r}$$

and

$$d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_s}.$$

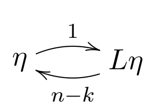
What kind of terms appear in $\Lambda L(dz_I \wedge d\bar{z}_J)$ or in $L\Lambda(dz_I \wedge d\bar{z}_J)$?

- $dz_{I+k-l} \wedge d\bar{z}_{J+k-l}$ for some $l \in I \cap J$, $k \notin I \cup J$, and
- $dz_I \wedge d\bar{z}_J$.

The coefficients of $dz_{I+k-l} \wedge d\bar{z}_{J+k-l}$ in $\Lambda L(dz_I \wedge d\bar{z}_J)$ and in $L\Lambda(dz_I \wedge d\bar{z}_J)$ match. The coefficient of $dz_I \wedge d\bar{z}_J$ in $(\Lambda L - L\Lambda(dz_I \wedge d\bar{z}_J)$ is $\#([n] - (I \cup J)) - \#(I \cap J) = n - \#(I \cup J) - \#(I \cap J) = n - \#I - \#J = n - k$. □

Let $\eta \in \mathcal{H}^k(X)$ be primitive. Then $(\Lambda L - L\Lambda)\eta = (n-k)\eta$, so $\Lambda L \eta = (n-k)\eta$.

So we have



and we want to build a string off to the right.

Define

$$\eta^{(r)} = \frac{L^r \eta}{r!} \quad \text{(might be 0)}.$$

Then we have

$$\eta = \eta^{(0)} \xrightarrow{1} \eta^{(1)} \xrightarrow{2} \eta^{(2)} \xrightarrow{3} \eta^{(3)} \xrightarrow{4} \cdots$$

Claim 5. $\Lambda\eta^{(r)} = (n-k-r+1)\eta^{(r-1)}$

Proof. Induction on r. We just checked the base case: $\Lambda(L\eta) = (n-k)\eta$.

Now we have

$$\begin{aligned}
\Lambda\eta^{(r)} &= \Lambda\frac{L^r\eta}{r!} = (n-k-2r+2)\frac{L^{r-1}\eta}{r!} + \frac{L\Lambda L^{r-1}\eta}{r!} \\
&= \frac{1}{r}((n-k-2r+2)\eta^{(r-1)} + L\Lambda\eta^{(r-1)} \\
&= \frac{1}{r}((n-k-2r+2)\eta^{(r-1)} + L(n-k-r)\eta^{(r-2)} \\
&= \frac{1}{r}((n-k-2r+2) + (r-1)(n-k-r+2))\eta^{(r-1)} \\
&= (n-k-r+1)\eta^{(r-1)}.
\end{aligned}$$

□

We have constructed

$$\eta^{(0)} \underset{n-k}{\overset{1}{\rightleftarrows}} \eta^{(1)} \underset{n-k-1}{\overset{2}{\rightleftarrows}} \cdots \underset{1}{\overset{n-k}{\rightleftarrows}} \eta^{(n-k)} \underset{0}{\overset{n-k+1}{\rightleftarrows}} \eta^{(n-k+1)} \underset{-1}{\overset{n-k+2}{\rightleftarrows}} \cdots$$

Corollary 6. *$\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(n-k)}$ are all nonzero. All higher $\eta^{(j)}$ are zero.*

Proof. If $j \le n-k$, then $\Lambda^j\eta^{(j)} = (\text{nonzero})\eta$, so $\eta^{(j)}$ is nonzero.

On the other hand, some $\eta^{(N)} = 0$. For $j > n-k$, $\eta^{(j)} = (\text{nonzero})\Lambda^{N-j}\eta^{(N)}$, so $\eta^{(j)} = 0$. □

Claim 7. *Suppose X is a compact Kähler manifold. Then, by induction on k, we can extend $H^0_{prim}, H^1_{prim}, \ldots, H^k_{prim}$ out to such strings, with $\bigoplus L^{\frac{m-j}{2}} H^j_{prim}$ injecting into H^m_{prim}.*

Proof. Say we've built such strings starting at $H^0_{\text{prim}}, H^1_{\text{prim}}, \ldots, H^{k-1}_{\text{prim}}$ Now $H^k(X) = \operatorname{Ker}\Lambda \oplus \operatorname{Im} L$, as $(L\alpha, \beta) = (\alpha, \Lambda\beta)$. All the strings we have built so far lie in $\operatorname{Im} L$, so the primitive cohomology is transverse to what we have already built.

If we had some relation $L^j\eta = \sum C_r L^j\theta_r$, $\eta \in \operatorname{Ker}\Lambda : H^k \to H^{k-2}$, and $\theta_r \in \operatorname{Im} L : H^{k-2} \to H^k$.

Apply Λ^j. On each string, $\Lambda^j L^j$ is some nonzero scalar, and we know there is no relation back in H^k_{prim}. □

Corollary 8. *If X is compact Kähler then*

$$b^0 \leq b^2 \leq b^4 \leq \cdots \leq b^{\,halfway}$$
$$b^1 \leq b^3 \leq b^5 \leq \cdots \leq b^{halfway}$$

Hodge star in terms of L and Λ

We know $L : H^k \xrightarrow{\cong} H^{2n-k}$ and $* : H^k \xrightarrow{\cong} H^{2n-k}$. What is the relationship?

Claim 9. *Let η be in $\mathcal{H}^{p,q}_{prim}$, meaning that $\Lambda\eta = 0$, and let $k = p+q$. Then*

$$*\eta = (-1)^{k(k+1)/2} i^{p-q} \frac{L^{n-k}}{(n-k)!}\eta = (-1)^{k(k+1)/2} i^{p-q} \eta^{(n-k)}$$

.

More generally, we define

$$\dagger = e^{-\Lambda} e^{L} e^{-\Lambda} : \mathcal{H}^{\bullet,\bullet} \to \mathcal{H}^{\bullet,\bullet}.$$

Note that L and Λ are nilpotent, so this operator makes sense.

We claim that, in general,

$$* = (-1)^{k(k+1)/2} i^{p-q} \dagger$$

on $\mathcal{H}^{p,q}$.

Let's see how this reduces to the claim. If η is primitive then

$$\begin{aligned} \dagger\eta &= e^{-\Lambda} e^{L} e^{-\Lambda\eta} \\ &= e^{-\Lambda} e^{L} (\eta + 0 + 0 + \cdots) \\ &= e^{-\Lambda} (\eta + \eta^{(1)} + \eta^{(2)} + \cdots + \eta^{(n-k)}) \\ &= (\text{stuff in degree} < (2n-k)) + \eta^{(n-k)} \end{aligned}$$

Since the above relation tells us that $\dagger\eta$ is a multiple of $*\eta$, we know that $\dagger\eta$ lives in degree $2n-k$. So all the terms in lower degree must be zero and we have $\dagger\eta = \eta^{(n-k)}$. Using the above relation again, we have

$$*\eta = (-1)^{k(k+1)/2} \; i^{p-q} \; \eta^{(n-k)}$$

as desired.

Now, we need to show

$$* = (-1)^{k(k+1)/2} i^{p-q} \dagger \,.$$

We claim this is true on $\Omega^{\bullet,\bullet}$, so we can check point by point in nice coordinates.

$\eta = \eta_1 \wedge \eta_2 \wedge \ldots \wedge \eta_n$, such that $\eta_j \in \{1, dz_j, d\bar{z}_j, A_j\}$, where $A_j = \frac{dz_j \wedge d\bar{z}_j}{-2i}$.

If L_j = wedge with A_j and Λ_j = contract with A_j then $L = \sum L_j$ and $\Lambda = \sum \Lambda_j$.

Note that, for $j \neq j'$, L_j and $L_{j'}$ commute, as do $L_j, \Lambda_{j'}$ and $\Lambda_j, \Lambda_{j'}$. So

$$\begin{aligned} e^{-\Lambda} e^{L} e^{-\Lambda} &= (e^{-\Lambda_1} \dots e^{-\Lambda_n})(e^{L_1} \dots e^{L_n})(e^{-\Lambda_1} \dots e^{-\Lambda_n}) \\ &= (e^{-\Lambda_1} e^{L_1} e^{-\Lambda_1}) \dots (e^{-\Lambda_n} e^{L_n} e^{-\Lambda_n}) \\ &= \dagger_1 \dots \dagger_n \,. \end{aligned}$$

So $\dagger\eta = (\dagger_1 \eta_1) \wedge \ldots \wedge (\dagger_n \eta_n)$, where $\dagger_j$ sends $1, A_j, dz_j$, and $d\bar{z}_j$ to $A_j, -1, dz_j$, and $d\bar{z}_j$, respectively. Up to a power of i, this is the same as $*\eta$.

Checking the sign (not done in class)

We now check the sign. Let a, b, c and d be the number of 1's, dz_j's, $d\overline{z}_j$'s and A_j's among the η's. For notational convenience, define δ_j by

$$\begin{array}{llll} \delta_j = & A_j & \text{if} \quad \eta_j = & 1 \\ & dz_j & & dz_j \\ & d\overline{z}_j & & d\overline{z}_j \\ & 1 & & A_j \end{array}$$

Let $\delta = \delta_1 \wedge \delta_2 \wedge \cdots \wedge \delta_n$.

Then $\dagger\eta = (-1)^d \delta$. (Just look at the signs in $e^{-\Lambda_j} e^{L} e^{-\Lambda_j}$.) Also, $*\eta = i^{c-b} (-1)^{\binom{b+c}{2}} \delta$. Here the i^{c-b} term is comes from the fact that $*$ turns dz_j into $-i dz_j$ and turns $d\overline{z}_j$ into $i d\overline{z}_j$; the $(-1)^{\binom{b+c}{2}}$ comes from the sign of rearranging $\eta \wedge \delta$ into $A_1 \wedge A_2 \wedge \cdots A_n$.

So

$$*\eta = i^{c-b} (-1)^{\binom{b+c}{2} + d} \eta.$$

We rewrite the coefficient as $i^{b-c} (-1)^{\binom{b+c}{2} + b + c + d}$. Note that $p - q = (b + d) - (c + d) = b - c$. So the first time is i^{p-q}. So we are left wanting to check that

$$(-1)^{\binom{b+c}{2} + b + c + d} = (-1)^{k(k+1)/2}.$$

We have $k = b + c + 2d$, so our goal is to show that

$$\frac{(b+c)(b+c-1)}{2} + b + c + d \equiv \frac{(b+c+2d)(b+c+2d+1)}{2} \mod 2.$$

This is what computer algebra systems are meant for. Subtracting the two sides, we need to check that

$$2(b+c+d)d \equiv 0 \mod 2$$

which is obvious.