NOTES FOR MARCH 31

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Let X be a compact Kähler manifold of dimension n, and define $L^{(n-k)} = \frac{L^{n-k}}{(n-k)!}$. Last time, we computed that $* : H^{p,q}_{\text{prim}}(X) \to H^{n-q,n-p}(X)$ is given by the formula $* = i^{p-q}(-1)^{\frac{k(k+1)}{2}} L^{(n-k)}$. Today we pursue consequences of this formula.

The positive definite Hermitian form on $H^n(X)$

Define a positive definite Hermitian form on $H^{p,q}_{\text{prim}}(X)$ by $(\alpha, \beta) = i^{p-q}(-1)^{\frac{k(k+1)}{2}} \int_X \alpha \wedge \overline{L^{(n-k)}\beta}$. This Hermitian form is in fact the hermitian form $\int_X \alpha \wedge *\overline{\beta}$ defined previously, but restricted to the primitive part.

We check that this is in fact positive definite in the following Corollary.

Corollary: For any primitive (p, q)-form η, $(\eta, \eta) \geq 0$, with equality if and only if $\eta = 0$.

Proof: Observe that $(\eta, \eta) = \int_X \eta \wedge \overline{i^{p-q}(-1)^{\frac{k(k+1)}{2}} L^{(n-k)}\eta} = \int_X \eta \wedge \overline{*\eta}$, and we know that $\int_X \eta \wedge \overline{*\eta} \geq 0$, with equality holding if and only if $\eta = 0$. □

Suppose that I know X and its complex structure, then I can decompose $H^k(X, \mathbb{C})$ into the direct sums of $H^{p,q}(X)$ without knowing the Kähler structure. (See March 22.) Let ω be the negative of the imaginary part of the Kähler form on X, and suppose all we know is what the class $[\omega] \in H^{1,1}(X)$ is. We will now show that this is enough information to recover the Hermitian positive definite pairing on $H^\bullet(X)$ given by $(\alpha, \beta) = \int_X \alpha \wedge *\overline{\beta}$. Note that since L commutes with Δ, we can view L as a map from $H^{p,q}(X) \to H^{p+1,q+1}(X)$, and this is given by $L : [v] \mapsto [\omega v]$. This allows us to say (by a problem in Problem Set 10) what the primitive cohomology is, so we can get the decomposition $H^{p,q}(X) = H^{p,q}_{\text{prim}}(X) \oplus LH^{p-1,q-1}_{\text{prim}}(X) \oplus L^2 H^{p-2,q-2}_{\text{prim}}(X) \oplus \dots$. Also, by knowing L, we can say what $(\cdot, \cdot)$ is on each $L^i H^{p-i,q-i}_{\text{prim}}(X)$, and since we know that the components of this decomposition are pairwise orthogonal under $(\cdot, \cdot)$, we can recover $(\cdot, \cdot)$ on all of $H^{p-q}(X)$.

Application – A theorem of Serre

Every graduate class should prove at least one publishable theorem. This one is Serre, "Analogues Kählériens de certaines conjectures de Weil", *Annals of Math.* **71** (1960).

Theorem: (Serre) If X is compact Kähler and $F : X \to X$ is an endomorphism such that $F^*[\omega] = q[\omega]$ for $q \in \mathbb{R}_{\geq 0}$, then the eigenvalues of F on $H^k(X)$ have norm $q^{\frac{k}{2}}$.

Proof: Since $F^*([\omega]) = q[\omega]$, we have that $F^*L([\eta]) = qLF^*([\eta])$, and in particular, F^* preserves $L^j H^{p-j,q-j}_{\text{prim}}(X)$. Also,

$$\begin{aligned} \int_X F^*\eta \wedge L^{(n-k)} F^*\overline{\eta} &= \frac{1}{q^{n-k}} \int_X F^*\eta \wedge F^*(L^{(n-k)}\overline{\eta}) \\ &= \frac{1}{q^{n-k}} \int_X F^*(\eta \wedge (L^{(n-k)}\overline{\eta})) \\ &= \frac{q^n}{q^{n-k}} \int_X \eta \wedge (L^{(n-k)}\overline{\eta}) \end{aligned}$$

The last equality holds because $[\eta \wedge L^{(n-k)}\overline{\eta}] \in H^{2n}(X)$, which is the $\mathbb{C}$-span of $[\omega^n]$. Thus, $[\eta \wedge L^{(n-k)}\overline{\eta}] = c[\omega^n]$ for some $c \in \mathbb{C}$, which means $F^*([\eta \wedge L^{(n-k)}\overline{\eta}]) = cF^*([\omega^n]) = q^n[\eta \wedge L^{(n-k)}\overline{\eta}]$.

The above computation shows that $(F^*\eta, F^*\eta) = q^k(\eta, \eta)$. (Note that F^* is $\mathbb{C}$-linear, so $F^*\overline{\eta} = \overline{F^*\eta}$.) Since the norm (,) is positive definite Hermitian, this implies that $q^{-\frac{k}{2}} F^*$ is unitary. So all eigenvalues of $q^{-k/2}F^*$ all have norm 1 and we see that eigenvalues of F^* all have norm $q^{k/2}$. □

Another Hermitian form on $H^n(X)$

Let $p+q=n-2j$, and consider $L^jH^{p,q}_{\text{prim}}(X)\subset H^{p+j,q+j}(X)\subset H^n(X)$. Define a positive definite Hermitian form on $L^jH^{p,q}_{\text{prim}}(X)$ given by $(\alpha,\beta)\mapsto i^{p-q}(-1)^{\frac{(p+q)(p+q+1)}{2}}\int_X\alpha\wedge\overline{\beta}$. This is positive definite because for any $\theta\in L^jH^{p,q}_{\text{prim}}(X)$, we can write $\theta=L^j\eta$ for $\eta\in H^{p-j,q-j}$, so $\int_X\theta\wedge\overline{\theta}=\int_X L^j\eta\wedge L^j\overline{\eta}=\int_X\eta\wedge L^{2j}\overline{\eta}$. Positive definiteness then follows from the relationship between $*$ and L, and the fact that $\int_X\alpha\wedge *\beta$ is a positive definite hermitian form.

On the other hand, we can avoid the complicated sign rule above and just define a Hermitian form on $H^n(X,\mathbb{C})$ by $\langle\alpha,\beta\rangle\mapsto\int_X\alpha\wedge\overline{\beta}$. Note that this is not positive definite.

Example: Consider the Kähler manifold $\mathbb{P}^1\times\mathbb{P}^1$, and let ω_1,ω_2 be the pullbacks of the Fubini-Study forms from each copy of $\mathbb{P}^1$. Then $H^\bullet(\mathbb{P}^1)=\mathbb{C}\oplus 0\oplus\mathbb{C}$, so $H^2(\mathbb{P}^1\times\mathbb{P}^1)=\mathbb{C}^2$ by Kunneth. Explicitly, the generators of H^2 are ω_1 and ω_2, the two Fubini-Study forms.

By computing the Fubini-Study metric in coordinates, we see that $\omega_1\wedge\omega_1=0$, so $\langle\omega_1,\omega_1\rangle=\int_X\omega_1\wedge\omega_1=\int_X 0=0$. Similarly, we can show that $\langle\omega_2,\omega_2\rangle=0$ and $\langle\omega_1,\omega_2\rangle=\langle\omega_2,\omega_1\rangle=1$.

Every $(1,1)$- form on $\mathbb{P}^1\times\mathbb{P}^1$ is of the form $a_1\omega_1+a_2\omega_2$ for a_1 and $a_2\in\mathbb{C}$. Thus, if $\omega=a_1\omega_1+a_2\omega_2$ is a Kähler form, then $\omega(u,v)>0$, which means $a_1\omega_1(u)+a_2\omega_2(v)>0$ for any nonzero (u,v). So the Kähler forms are $\omega=a_1\omega_1+a_2\omega_2$ for a_1 and $a_2>0$.

Using the Lefschetz decomposition, we can write $H^{1,1}(X)=H^{1,1}_{\text{prim}}(X)\oplus LH^{0,0}_{\text{prim}}(X)$, and it is easy to see that $LH^{0,0}_{\text{prim}}$ is the $\mathbb{C}$-span of ω while $H^{1,1}_{\text{prim}}$ is the $\mathbb{C}$-span of $\theta:=a_1\omega_1-a_2\omega_2$. A straight forward computation then gives us that $\langle\omega,\omega\rangle=2a_1a_2$ and $\langle\theta,\theta\rangle=-2a_1a_2$, so $\langle\cdot,\cdot\rangle$ is positive definite on $LH^{0,0}_{\text{prim}}$ and negative definite on $H^{1,1}_{\text{prim}}$.

Note that $\langle\cdot,\cdot\rangle$ is $\mathbb{R}$-valued when n is even and imaginary valued when n is odd. Also, when n is even, the restriction of $\langle\cdot,\cdot\rangle$ to $H^n(X,\mathbb{R})$ is $\mathbb{R}$-valued, and so it is a symmetric pairing for elements in $H^n(X,\mathbb{R})$. Recall that, for any symmetric bilinear form over $\mathbb{R}$ (or Hermitian bilinear form over $\mathbb{C}$), the signature of the form is the difference between the number of $+1$'s and number of -1's when you diagonalize the form.

We can compute the signature of $\langle\ ,\ \rangle$ on $H^n(X)$ by counting up the dimensions of the spaces $L^jH^{p-j,q-j}_{\text{prim}}(X)$ (with $p+q=n$) on which we know $\langle\ ,\ \rangle$ to be positive or to be negative.

Working out explicitly what $i^{p-q}(-1)^{k(k+1)/2}$ is, we can read off the sign of $\langle\ ,\ \rangle$ on $L^jH^{p-j,q-j}(X)$ in the following top half diamond:

$$
\begin{array}{lccccccccc}
H^0(X) & & & & & + & & & & \\
H^2(X) & & & & + & - & + & & & \\
H^4(X) & & & + & - & + & - & + & & \\
H^6(X) & & + & - & + & - & + & - & + & \\
H^8(X) & + & - & + & - & + & - & + & - & + \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

For example, the sign on $L^jH^{1,1}_{\text{prim}}$ is negative, and the sign on $L^jH^{4,2}(X)$ is negative.

The special case of $H^{1,1}$ on a surface

On a surface (complex dimension 2), we can write $H^2=H^{2,0}\oplus H^{1,1}\oplus H^{0,2}$ and $[\omega]\in H^{1,1}$. $LH^{0,0}_{\text{prim}}$ is $\mathbb{C}\cdot\omega$. By considering the diamond above, we see that $\langle\cdot,\cdot\rangle$ is negative definite on $H^{1,1}_{\text{prim}}(X)$.

For $\eta\in H^{1,1}(X)$, let $a=\langle\omega,\eta\rangle\,/\,\langle\omega,\omega\rangle$. The orthogonal projection of η onto $H^{1,1}_{\text{prim}}(X)$ is $[\eta-a\omega]$. So $\langle\eta-a\omega,\eta-a\omega\rangle\le 0$. Rearranging terms in this sum gives

$$\langle\eta,\eta\rangle\,\langle\omega,\omega\rangle\le\langle\omega,\eta\rangle^2$$

with strict inequality if $[\eta]$ is not a multiple of $[\omega]$. For those who know the algebraic picture, compare to Hartshorne Theorem V.1.9.