HODGE THEORY FOR $\mathbb{R}$- MANIFOLDS

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1. Vector bundles with an inner product, and $*$

Let X be an $\mathbb{R}$-fold, and let $\pi : E \longrightarrow X$ be a real vector bundle, of rank r, equipped with a positive definite symmetric bilinear form.

If $e_1, \dots, e_r \in \pi^{-1}(X)$ are orthonormal, then $e_1 \wedge \cdots \wedge e_r$ is a non-trivial vector in $\bigwedge^r E$.

Proposition: If $f_1, \dots, f_r$ is any other orthonormal basis for $\pi^{-1}(X)$, then $e_1 \wedge \cdots \wedge e_r = \pm f_1 \wedge \cdots \wedge f_r$.

Proof. Note that $f_i = g \cdot e_i$ for $g \in O(r)$, so $det(g) = \pm 1$. □

So, given this data, I get two points in each fiber of $\bigwedge^r E$.

Assumption: This is a trivial two fold cover. Choose one connected component and call that section of $\bigwedge^r E$ τ. Given this data, I have:

$$\bigwedge^r E \times \bigwedge^r E \longrightarrow \mathbb{R}$$
$$\bigwedge^k E \times \bigwedge^{r-k} E \longrightarrow \bigwedge^r E \longrightarrow \mathbb{R}$$

First Pairing:

The inner product on E gives a map $E \longrightarrow E^*$, so functorially I get

$$\bigwedge^k E \longrightarrow \bigwedge^k (E^*) \cong (\bigwedge^k E)^*.$$

Concretely, given vectors $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k \in \bigwedge^k E$, we have

$$\langle v_1 \wedge \cdots \wedge v_k,\ w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle),$$

extended linearly. This is a symmetric pairing.

Second Pairing:

$$\bigwedge^k E \times \bigwedge^{r-k} E \longrightarrow \bigwedge^r E \longrightarrow \mathbb{R}$$

Lemma: Both of these are perfect pairings.

Proof. Check in an orthonormal trivialization. □

So, we can get a map

$$* : \bigwedge^k E \longrightarrow \bigwedge^{r-k},$$

making these two pairings coincide.

In an orthonormal trivialization, where $\tau = e_1 \wedge \cdots \wedge e_n$

$$* : e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto (-1)^{\sigma} e_{j_1} \wedge \cdots \wedge e_{j_{r-k}}$$

where $1 \le i_1 < i_2 < \cdots < i_k \le r$ and $1 \le j_1 < j_2 < \cdots < j_{r-k} \le r$ with

$$[r] = \{i_1, \dots, i_k\} \sqcup \{j_1, \dots, j_{r-k}\}$$
$$\sigma = |\{(i_a, j_b) \| j_b < i_a\}|.$$

Thus, $** = (-1)^{k(r-k)}\mathrm{Id}$.

In a general basis $v_1, \dots, v_r$, let $g_{ij} = \langle v_i, v_j \rangle$, then

$$\det(g) = \langle v_1 \wedge \cdots \wedge v_r, v_1 \wedge, \dots, \wedge v_r \rangle$$
$$\tau = \pm \frac{v_1 \wedge \cdots \wedge v_r}{\sqrt{\det(g)}}$$

and

$$* : v_{i_1} \wedge \cdots \wedge v_{i_k} \mapsto \sum_{J'} (-1)^{\sigma(I',J')} \frac{\det(g_{i_a i'_b})}{\sqrt{\det(g)}} v_{j'_1} \wedge \cdots \wedge v_{j'_{r-k}},$$

with $I' = [r]\ J'$.

Note: The $*$ map is defined fiber by fiber. In other words, it is C^∞-linear.

2. $*$ on the tangent bundle

Let E be T_*X.

Given an inner product on E, we can get an inner product on E^*. In coordinates, if $\langle v_i, v_j\rangle = g_{ij}$, then $\langle v_i^*, v_j^*\rangle = (g^{-1})_{ij}$. So we may equivalently talk about an inner product on T^*X.

Note: Any E has some inner product on it. Any X has some inner product on T^*X.

Side comment: Why is differential geometry so hard? Whenever doing computations in E, a vector bundle with an inner product, you want to work in an orthonormal basis. Whenever you are doing computations in T_*X, you want to work in $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. You can't do both!

Let X be a smooth, oriented, real n-fold with a positive definite symmetric bilinear form on T_*X, and hence on T^*X.

"Oriented" deals with τ : we'll have $\tau = e_1 \wedge \cdots \wedge e_n$, where e_i is an orthonormal basis respecting the orientation. So, we get $* : \bigwedge^k T^*X \longrightarrow \bigwedge^{n-k} T^*X$ and we get a map of sheaves of $\mathcal{C}^\infty$-modules from $\Omega^k \longrightarrow \Omega^{n-k}$. As before, $** = (-1)^{k(n-k)}$.

If X is compact, we get an inner product on $\Omega^k(X)$:

$$(\alpha, \beta) := \int_X \alpha \wedge *\beta = \int_X \langle \alpha, \beta\rangle \cdot \tau$$

Notice that $(\alpha, \beta) = (\beta, \alpha)$ because $\langle\ ,\ \rangle$ is symmetric.

Also,

$$(*\alpha, *\beta) = \int_X *\alpha \wedge **\beta = (-1)^{k(n-k)} \int_X *\alpha \wedge \beta = \int_X \beta \wedge *\alpha = (\beta, \alpha) = (\alpha, \beta).$$

Finally $(\alpha, \alpha) \geq 0$, with equality iff $\alpha = 0$, as $\langle, \rangle$ is a positive definite pairing and τ is everywhere positive w.r.t the orientation on X. So (,) is a positive definite, non-degenerate, symmetric bilinear form on $\Omega^k(X)$.

We now define $d^* : \Omega^k \longrightarrow \Omega^{k-1}$ as

$$(-1)^k *^{-1} d* = (-1)^{k+(k-1)(n-k+1)} * d * .$$

When n is even, we get $d^* = - * d *$.

Note: d^* is defined locally.

Key Fact: Let X be compact. For $\alpha \in \Omega^{k-1}$, $\beta \in \Omega^k$, we have

$$(d\alpha, \beta) = (\alpha, d^*\beta)$$

Proof. By definition

$$(d\alpha, \beta) = \int_X d\alpha \wedge *\beta$$

Also,

$$(\alpha, d^*) = \int_X \alpha \wedge (-1)^k **^{-1} d * \beta = (-1)^k \int_X \alpha \wedge d(*\beta)$$

Now,

$$\int d\alpha \wedge (*\beta) + (-1)^{k-1} \int \alpha \wedge d(*\beta) = \int d(\alpha \wedge *\beta) = 0,$$

by Stokes' theorem. □

Remarks:

d does not see the metric.

$*$ sees the metric, but works fiber by fiber.

d^* sees the metric and its derivatives, and the derivatives of your k-form

(,) is honestly global.

3. The Laplacian and the Hodge theorem

We define the Laplacian $\Delta_d : \Omega^k \longrightarrow \Omega^k$ by

$$\Delta_d := dd^* + d^*d$$

Note that if $f, g \in \Omega^k(X)$, then

$$\begin{aligned}(f, \Delta_d g) &= (f, dd^*g) + (f, d^*dg) \\ &= (d^*f, d^*g) + (df, dg).\end{aligned}$$

In particular, $(f, \Delta_d f) \geq 0$ and we have equality iff $df = 0$ and $d^*f = 0$. So, we get a map

$$\mathrm{Ker}(\Delta : \Omega^k(X) \longrightarrow \Omega^k(X)) \longrightarrow \{\text{closed k-forms}\} \twoheadrightarrow \mathrm{H}^k_{DR}(X).$$

Theorem (Hodge):

$$\mathrm{Ker}(\Delta : \Omega^k(X) \longrightarrow \Omega^k(X)) \cong \mathrm{H}^k_{DR}(X).$$

Injectivity is not hard (problem set 7, problem 2), but surjectivity is!

We won't prove this. Instead, I'll give you some intuition, and I'll tell you what is true.

4. The intuition for the Hodge theorem

Let

$$0 \to V^0 \to V^1 \to \cdots \to V^n \to 0$$

be a complex of finite dimensional vector spaces over $\mathbb{R}$.

Let $(\quad , \quad)$ be a positive definite inner product on V^i. Then we can take about the adjoint map $d^* : V^k \longrightarrow V^{k-1}$, defined so that

This data gives maps $d^* : V^k \longrightarrow V^{k-1}$ with $(f, dg) = (d^*f, g)$.

Define $\Delta := dd^* + d^*d$.

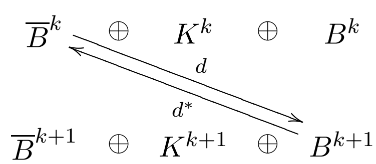
Theorem: $\mathrm{Ker}(\Delta : V^k \longrightarrow V^k) \cong \mathrm{H}^k(V^\bullet) = \frac{\mathrm{Ker}(d : V^k \longrightarrow V^{k+1})}{\mathrm{Im}(d : V^{k-1} \longrightarrow V^k)}$

Proof. Let $Z^k = \mathrm{Ker}(d : V^k \longrightarrow V^{k+1})$ and $B^k = \mathrm{Im}(d : V^{k-1} \longrightarrow V^k)$.

Thus, $B^k \subset Z^k \subset V^k$.

Let K^k be the orthogonal component to B^k in Z^k. Let $\overline{B}^k$ be the orthgonal component to Z^k in V^k. So, $V^k = B^k \oplus \overline{B}^k \oplus K^k$.

Thus, d is 0 on B^k and K^k and is injective on $\overline{B}^k$. If we consider d as a map $\overline{B}^k \to B^{k+1}$, then it is an isomorphism, as $\overline{B}^k$ is tranverse to the kernel of d, and B^{k+1} is the image of d.



Since this direct sum decompositions are orthogonal, the map d^* is 0 on $\overline{B}^{k+1}$ and K^{k+1}, and maps to $\overline{B}^k$. The map $d^* : B^{k+1} \to \overline{B}^k$ is the adjoint of an isomorphism, so it is an isomorphism.

On $\overline{B}^k$, we have $\Delta = d^*d$, the composition of two isomorphisms, so it is an isomorphism. On B^k, apply the same argument between ranks k and $k-1$ to see that $\Delta = dd^*$ is an isomorphism. On K^k, both d and d^* vanish, so Δ is 0 there.

So

$$\mathrm{Ker}(\Delta) = K^k \cong \frac{K^k \oplus B^k}{B^k} = \frac{Z^k}{B^k} =: H^k(V^\bullet)$$

□

Notice that $(\Delta f, g) = (f, \Delta g)$ so Δ is self-adjoint and is, thus diagonalizable. Also, $(\Delta f, f) = (df, df) + (d^*f, d^*f) \geq 0$, so Δ is positive semidefinite, and its eigenvalues must be ≥ 0. We just saw that 0-eigenspace of Δ on V^k is isomorphic to $\mathrm{H}^k(V^\bullet)$.

In Problem Set 7, Problem 3, you will check that d and d^* take the λ-eigenspace of V^k to the λ-eigenspace of $V^{k\pm1}$, so $V^\bullet = \bigoplus V^\bullet_\lambda$. You will see that $V^\bullet_\lambda$ is exact for $\lambda > 0$.

5. The truth

There is a sequence of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_i \to \infty$ such that $\Delta : \Omega^k(X) \longrightarrow \Omega^k(X)$ has a finite dimensional λ_i-eigenspace. Every k-form is an infinite sum of eigenfunctions

$$\eta = \sum w_i$$

with the sum and all its derivatives uniformly, absolutely convergent.

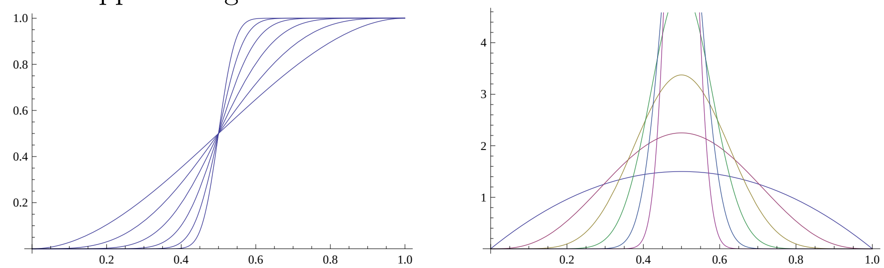
The operators d and d^* take the λ-eigenspace to the λ-eigenspace.

On the 0-eigenspace, d and d^* are zero. For $\lambda > 0$, the resulting complex is exact.

6. Appendix: Why is the Hodge Theorem hard? Added by David, and purely for the curious

The vector space $\Omega^k(X)$ is infinite dimensional. If $\Omega^k(X)$ was a Hilbert space, and d and d^* were bounded operators, this wouldn't be too bad.

Let's try putting the L^2-norm on $\Omega^k(X)$. Since we've already chosen a norm, this is at least canonical. Now, $\Omega^k(X)$ isn't complete in the L^2 norm. The left hand graph below shows a family of smooth functions approaching a discontinuous function in the L^2 norm.



We could complete $\Omega^k(X)$ in the L^2 norm. But differential operators don't extend continuously to L^2! The right hand side of the figure shows the derivatives of the functions on the left. The height of the bump is growing like N, even as the width shrinks like $1/N$, so the L^2 norm is going to infinity like N, and there is no L^2 limit for the sequence of derivatives.

What we do instead is to create a whole sequence of topologies on $\Omega^k(X)$, called the Sobolev topologies. A sequence of smooth functions is convergent in the s-th Sobolev topology if all of its derivatives up to order s are L^2-convergent. The map Δ then decreases s by 2. One must do functional analysis on these spaces in order to establish that Δ has the required eigenvalue structure there. One then must prove that the resulting eigenfunctions actually are smooth k-forms.