Problem 1 Let $G = \operatorname{Spec} k[u, u^{-1}]$ and let μ be the map $G \times G \to G$ corresponding to the map of rings $k[u, u^{-1}] \to k[u_1, u_1^{-1}, u_2, u_2^{-1}]$ which sends u to $u_1 u_2$. Let $\iota : \operatorname{Spec} k \to G$ correspond to the map of k-algebras $k[u, u^{-1}] \to k$ sending $u \mapsto 1$. By action of G on a scheme X, we mean a map of schemes $\alpha : G \times X \to X$ so that the two obvious maps $G \times G \times X \to X$ are equal and such that the composition $X \cong \operatorname{Spec} k \times_k X \xrightarrow{\iota \times \mathrm{Id}} G \times X \xrightarrow{\alpha} X$ is the identity.

Let S be a k-algebra. In this problem, we will see that $\mathbb{Z}$-gradings on S correspond to actions of G on $\operatorname{Spec} S$.

(a) Let $S = \bigoplus_{j=-\infty}^{\infty} S_j$ be a grading. Define a map $\alpha^* S \to S[u, u^{-1}]$ by $\alpha^*(f) = u^j f$ for $f \in S_j$. Show that α^* is a map of rings and the induced map $G \times \operatorname{Spec} S \to \operatorname{Spec} S$ is an action.

Let $\alpha : G \times \operatorname{Spec} S \to \operatorname{Spec} S$ be any map of schemes and let $\alpha^* : S \to S[u, u^{-1}]$ be the corresponding map of rings. Define $S_j = \{f \in S : \alpha^* f = u^j f\}$.

(b) Show that, if $f_j \in S_j$ and $\sum f_j = 0$ then each f_j is 0.

(c) Show that, if $f_i \in S_i$ and $f_j \in S_j$ then $f_i f_j \in S_{i+j}$.

Now, assume that α is an action.

(d) For any $f \in S$, let $\alpha^* f = \sum u^j f_j$. (Note that this is a finite sum.) Show that $f_j \in S_j$ and $f = \sum f_j$.

(e) Explain why we are done, i.e., why we have shown that an action of G on $\operatorname{Spec} S$ gives a $\mathbb{Z}$-grading of S.

Problem 2 Let S be a $\mathbb{Z}$-graded ring. Let $\operatorname{Homog}(S)$ be the set of homogenous primes of S. For a positive integer d, define $S^{(d)}$ to be the subring $\bigoplus_{j \geq 0} S_{jd}$ of S. In this problem, we will check some basic compatibilities between these rings. I've broken this into a lot of parts; they are all meant to be short.

(a) Let I be a homogenous ideal of S. Show that I is prime if and only if, for any homogenous elements f and g, if $fg \in I$, then either $f \in I$ or $g \in I$.

(b) Let I be a homogenous ideal of S. Show that $\sqrt{I} := \{f \in S : \ f^n \in I \text{ for some } n > 0\}$ is a homogenous ideal.

(c) Let $\mathfrak{p}$ be a homogenous prime ideal of S. Show that $\mathfrak{p} \cap S^{(d)}$ is a homogenous prime ideal of $S^{(d)}$.

(d) Let $\mathfrak{q}$ be a homogenous prime ideal of $S^{(d)}$. Show that $\sqrt{\mathfrak{q} S}$ is a homogenous prime ideal of S.

(e) Show that (c) and (d) provide inverse bijections between $\operatorname{Homog}(S)$ and $\operatorname{Homog}(S^{(d)})$.

(f) Let $f \in S_d$. Provide bijections between the following sets: $\{\mathfrak{p} \in \operatorname{Homog}(S) : \ f \not\in \mathfrak{p}\}$, $\{\mathfrak{q} \in \operatorname{Homog}(S^{(d)}) : \ f \not\in \mathfrak{q}\}$, $\operatorname{Homog}(f^{-1} S^{(d)})$, $\operatorname{Spec}((f^{-1} S^{(d)})_0)$, $\operatorname{Spec}((f^{-1} S)_0)$.

The final composite bijection between $\{\mathfrak{p} \in \operatorname{Homog}(S) : \ f \not\in \mathfrak{p}\}$ and $\operatorname{Spec}((f^{-1} S)_0)$ is justified by Hartshorne under the statement "The properties of localization show that ϕ is bijective …" in the proof of Proposition 2.5; I wrote this exercise to understand that sentence. If you see a faster route, let me know!