Here are some more problems about properness:

Problem 1.(a) Let k be an algebraically closed field and let X be a connected, reduced, proper scheme over k. Show that $\mathcal{O}(X) = k$. (Hint: Identify $\mathcal{O}(X)$ with $\mathrm{Hom}(X, \mathbb{A}^1_k)$, then embed $\mathbb{A}^1_k$ into $\mathbb{P}^1_k$.

(b) Let X be as above and let U be an affine k-scheme of finite type. Show that all maps from X to U are constant.

(c) Suppose that we remove the hypothesis that k is algebraically closed (but X is still a connected, reduced, proper scheme over k). Show that $\dim_k \mathcal{O}(X) < \infty$.

Problem 2 In this problem, we will prove the rigidity lemma. Let k be an algebraically closed field (there are versions without algebraic closure, but I'm trying to keep this simple). Let X, B and Z be reduced k-schemes of finite type over k. Suppose that X is connected, proper over k, and nonempty; that Z is separated and B is connected. Let $\phi : B \times X \to Z$ be a morphism of k-schemes. Let b be a point of B with fraction field k, let z be a point of Z and suppose that $\phi : X \times \{b\} \to Z$ is the constant map to z. We will show that, for every $b' \in B$, the map $\phi : X \times \{b'\} \to Z$ is a constant map.

(a) Let U be an affine open neighborhood of z in Z. Show that there is an open neighborhood C of $b \in B$ such that $\phi(X \times C) \subseteq U$.

(b) Show that, for every $b' \in C$, the map $\phi : X \times \{b'\} \to U$ is constant. (Hint: Think about problem 1.)

(c) Choose a point $x_0 \in X$ with fraction field k and let ι be the inclusion of x_0 into X. Let π_2 be the projection $X \times B \to B$ and define $\psi : X \times B \to \phi$ to be $\phi \circ (\iota \times \pi_2)$. (In other words, on the level of closed points, $\psi(x, b) = \phi(x_0, b)$.) Show that $\phi = \psi$ everywhere on B. (Hint: B is connected.)

Problem 3 Here is a fun application of the rigidity lemma: Let G be a connected, proper algebraic group over a field k. Show that G is abelian. (Hint: Think about the map $G \times G \to G$ given by $(g_1, g_2) \mapsto g_1 g_2 g_1^{-1} g_2^{-1}$.)