Hints for this problem set are on the rear of the page. I think this is pretty hard, but doable, without them.

**Problem 1** In this problem, we'll investigate the relationship between partitions of unity and sheaf cohomology vanishing. Let $(X, \mathcal{O})$ be a locally ringed space.

We say that $(X, \mathcal{O})$ and let $U_i$ be a locally finite cover of $U_i$. We define a ***strong partition of unity*** subordinate to $U_i$ to be functions $\phi_i \in \Gamma(\mathcal{O})$ such that $\sum \phi_i = 1$ and, if $z \in X \setminus U_i$, then $\phi_i$ vanishes in the stalk $\mathcal{O}_z$. We will say that $\phi_i$ is a ***weak partition of unity*** subordinate to $U_i$ if the same holds but we only require $\phi_i$ to lie in the maximal ideal $\mathfrak{m}_z \subset \mathcal{O}_z$. (These are not standard terms.) Clearly, strong partitions of unity implies weak partitions of unity.

(a) Suppose that $X$ is paracompact (every open cover has a locally finite refinement) and regular (for any point $z$ and any closed set $K$, there is are open sets $V \ni z$ and $W \supset K$ with $V \cap W = \emptyset$). Suppose that every locally finite cover of $X$ has a weak partition of unity. Show that every open cover of $X$ has a locally finite refinement with a strong partition of unity.

From now on, suppose that every open cover of $X$ has a locally finite cover with respect to which there is a strong partition of unity.

(b) Suppose that $\mathcal{B} \to \mathcal{C}$ is a surjection of sheaves of $\mathcal{O}$-modules. Show that $\Gamma(\mathcal{B}) \to \Gamma(\mathcal{C})$ is a surjection.

(c) Let $\mathcal{A}$ be an arbitrary sheaf of $\mathcal{O}$-modules. Show that $H^1(\mathcal{A}) = 0$. You may assume that $\mathcal{A}$ injects into an injective $\mathcal{O}$-module $\mathcal{I}$.

(d) Let $\mathcal{A}$ be an arbitrary sheaf of $\mathcal{O}$-modules. Show that $H^q(\mathcal{A}) = 0$. You may assume that every sheaf of $\mathcal{O}$-modules injects into an injective sheaf of $\mathcal{O}$-modules.

**Problem 2** Given a map of complexes $A^\bullet \xrightarrow{f} F = B^\bullet$, define the cone complex $\mathrm{Cone}(A^\bullet \xrightarrow{f} B^\bullet)$ where $\mathrm{Cone}(A^\bullet \to B^\bullet)^q = A^{q+1} \oplus B^q$ and where the map $A^{q+1} \oplus B^q \to A^{q+2} \oplus B^{q+1}$ is given by $\left(\begin{smallmatrix} \partial & (-1)^q f \\ 0 & \partial \end{smallmatrix}\right)$. We have obvious maps of complexes $B^\bullet \xrightarrow{i} \mathrm{Cone}(A^\bullet \xrightarrow{f} B^\bullet) \xrightarrow{j} A^{\bullet+1}$.

(a) Abbreviating $\mathrm{Cone}(A^\bullet \xrightarrow{f} B^\bullet)$ to $K^\bullet$, show that we have a long exact sequence

$$\cdots \to H^0(B^\bullet) \xrightarrow{i^*} H^0(K^\bullet) \xrightarrow{j^*} H^1(A^\bullet) \xrightarrow{f^*} H^1(K^\bullet) \xrightarrow{i^*} H^2(A^\bullet) \to \cdots,$$

where we have labeled each arrow by the map of complexes which induces it. Note that $H^q(A^{\bullet+1}) = H^{q+1}(A^\bullet)$, so $j^*$ maps $H^q(K^*) \to H^{q+1}(A^\bullet)$.

Now suppose that we have three complexes $A^\bullet$, $B^\bullet$ and $C^\bullet$ and maps of complexes $A^\bullet \xrightarrow{\alpha} B^\bullet$ and $B^\bullet \xrightarrow{\beta} C^\bullet$ such that $0 \to A^q \xrightarrow{\alpha} B^q \xrightarrow{\beta} C^q \to 0$ is exact for all $q$.

(b) Construct a map of complexes $\mathrm{Cone}(A^\bullet \xrightarrow{\alpha} B^\bullet) \to C^\bullet$ which induces isomorphisms $H^q(\mathrm{Cone}(A^\bullet \xrightarrow{\alpha} B^\bullet)) \cong H^q(C^\bullet)$ for all $q$.

Combining the two problems, we have constructed a long exact sequence

$$\cdots \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to H^1(B^\bullet) \to H^1(C^\bullet) \to$$

Hint for 1(a): Let our cover be $X = \bigcup U_i$ and let $K_i = X \setminus U_i$. For every $z \in X$, choose a $U_i \ni z$ and choose disjoint opens $V_z \ni z$ and $W_z \supset K_i$. Consider show that weak partitions of unity subordinate to a locally finite refinement of $V_z$ are strong partitions of unity with respect to $U_i$.

Hint for 1(b): Lift $c \in \Gamma(\mathcal{C})$ to $b_i \in \mathcal{B}(U_i)$. After refining appropriately, take a strong partition of unity $\phi_i$ subordinate to $U_i$. Show that $\phi_i b_i$ extends to an element of $\Gamma(\mathcal{B})$, and that $\sum \phi_i b_i \mapsto c$.

Hint for 1(c): Use part (b) and play with the injection $\mathcal{A} \to \mathcal{I}$.

Hint for 1(d): Play with the injection $\mathcal{A} \to \mathcal{I}$.