Problem 1 This problem explores why we use rightward, not leftward, acyclic resolutions when working with sheaf cohomology.

Let X be a topological space, and let

$$0 \to \mathbb{F}_n \to \mathbb{F}_{n-1} \to \cdots \to \mathbb{F}_0 \to \mathbb{E} \to 0$$

be an exact sequence of sheaves of abelian groups on X. Suppose that, for all q greater than 0, and all i, we have $H^q(X, \mathbb{F}_i) = 0$. Show that

(a) For $q > 0$, we have $H^q(X, \mathbb{E}) = 0$.

(b) The sequence

$$0 \to H^0(X, \mathbb{F}_n) \to H^0(X, \mathbb{F}_{n-1}) \to \cdots \to H^0(X, \mathbb{F}_0) \to H^0(X, \mathbb{E}) \to 0$$

is exact.

Problem 2 In this problem, we will explore some examples of Čech hypercohomology. Let $\mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to \cdots$ be a complex of sheaves on X and let U_i be an open cover of X. Then we define the Čech double complex of $\mathcal{E}^\bullet$ to be the double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{i_0<i_1} \mathcal{E}^0(U_{i_0} \cap U_{i_1}) & \longrightarrow & \bigoplus_{i_0<i_1} \mathcal{E}^1(U_{i_0} \cap U_{i_1}) & \longrightarrow & \bigoplus_{i_0<i_1} \mathcal{E}^2(U_{i_0} \cap U_{i_1}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_{i_0} \mathcal{E}^0(U_{i_0}) & \longrightarrow & \bigoplus_{i_0} \mathcal{E}^1(U_{i_0}) & \longrightarrow & \bigoplus_{i_0} \mathcal{E}^2(U_{i_0}) & \longrightarrow & \cdots
\end{array}$$

where the horizontal maps come from the complex $\mathcal{E}^\bullet$ and the vertical maps come from restriction. We'll write $\mathcal{C}^{p,q}$ for the (p,q) term, $\bigoplus_{i_0<i_1<\cdots<i_q} \mathcal{E}^p(U_{i_0} \cap \cdots \cap U_{i_q})$. Set $\mathcal{C}^k = \bigoplus_{p=0}^{k} \mathcal{C}^{p,k-p}$ and make $\mathcal{C}^\bullet$ into a complex in the standard way. (Specifically, the map $\mathcal{C}^{p,k-p} \to \mathcal{C}^{p,k-p+1}$ is the vertical map in the double complex and $\mathcal{C}^{p,k-p} \to \mathcal{C}^{p+1,k-p}$ is $(-1)^{k-p}$ times the horizontal map in the double complex. See `http://ncatlab.org/nlab/show/total+complex` for more and don't worry too much about signs.)

If the $\mathcal{E}$ are quasi-coherent and all the overlaps $U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_p}$ are affine, then $\mathbb{H}^k(\mathcal{E}^\bullet) \cong H^k(\mathcal{C}^\bullet)$.

(a) Let k be a field of characteristic 0. Compute $\mathbb{H}(\mathcal{O} \xrightarrow{d} \Omega^1)$ on $\mathbb{P}^1_k$.

(b) Let k be a field of characteristic 0. Compute $\mathbb{H}(\mathcal{O} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2)$ on $\mathbb{A}^2_k \setminus \{(0,0)\}$.

Hint for both parts: The double complex breaks into a direct sum of double complexes of finite dimensional vector spaces, using gradings on the rings.