ALGEBRAIC GEOMETRY II – THE DAILY UPDATE

Contents

January 7 – Introduction. In this course, we will learn to work with schemes. (We will also learn sheaf cohomology, but more about that after March Break.) I'm going to steal a brilliant comment by Allen Knutson on Mathoverflow[1].

> One of the wholly unnecessary reasons that schemes are regarded with such fear by so many mathematicians in other fields is that three, largely orthogonal, generalizations are made simultaneously.
>
> Considering a "variety" to be Spec or Proj of a domain finitely generated over an algebraically closed field, the generalizations are basically
>
> (1) Allowing nilpotents in the ring.
> (2) Gluing affine schemes together.
> (3) Working over a base ring that isn't an algebraically closed field (or even a field at all).

As Ravi Vakil mentions in the comments, we should also add

> (4) Working with prime ideals, rather than maximal ideals.

We saw the pain of not being able to talk about nilpotents when we talked about Bezout's theorem, and had to talk about counting intersection points with multiplicity, or when we had to distinguish between naive and scheme theoretic fiber length. With schemes, we will be natively allowed to think about nilpotent elements of rings as functions on schemes. The downside is that we will have to give up on checking equality of functions point by point: A nilpotent function on a scheme X is zero at every point of X, but is not the zero function.

With schemes, we will no longer have to embed our spaces in affine or projective space; we will be able to directly glue two schemes together to form another scheme. This does give us some new schemes – not all schemes are quasi-projective. And non-quasi-projective schemes are not bizarre or pathological, they really do come up. That said, they are a bit of a specialty : I would guess that 90 percent of algebraic geometry papers or talks include no such examples. The real gift here is the freedom to talk about the glued object before proving that it is quasi-projective. We can directly build the Grassmannian by gluing together linear charts; we can define the normalization of a variety by normalizing each chart; we can talk about tangent and cotangent bundles by gluing together local trivializations.

Working with a non-algebraically closed base field is obviously important in number theory: If we want to talk about rational points on elliptic curves (say), we need to be able to think of those elliptic curves are schemes over the field $\mathbb{Q}$. The curves $x^3 + y^3 = z^3$ and $x^3 + y^3 = 9z^3$ are isomorphic over $\mathbb{C}$, but the first has only a few trivial points over $\mathbb{Q}$ and the second has infinitely many. More importantly, we want to be able to make geometric constructions and be able to talk about the results of those constructions as schemes over $\mathbb{Q}$ once again. All of that said, I should acknowledge that, rather than working with schemes over $\mathbb{Q}$, I often find it easier to think over $\overline{\mathbb{Q}}$ and keep track of the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ action.

Finally, the switch from maximal ideals to prime ideals. This is forced on us by functoriality: Think about the inclusion $k[x] \hookrightarrow k(x)$. The maximal ideal (0) of $k(x)$ pulls back to a prime ideal of $k[x]$, not a maximal ideal. This is very important for number theorists: We want to be able to take a curve X over $\mathbb{Q}$ and switch view points to think about a surface over the base $\operatorname{Spec} \mathbb{Z}$. The rational points on X then give rise to entire curves in $\mathcal{X}$. If we only talked about maximal ideals, we wouldn't be able to talk about the rational points of X

[1] http://mathoverflow.net/questions/28496/what-should-be-learned-in-a-first-serious-schemes-course/28594#28594

once we switched viewpoints to study $\mathcal{X}$. There are also a number of places in more classical settings where thinking in terms of prime ideals provides an extra conceptual clarity. I'll try to point these examples out when they arise.

January 9 – Concrete sheaves. Today we introduced the notion of a ***concrete sheaf***.

Definition. Let X be a topological space, E a set. A sheaf of E-valued functions on X is the data, for every open set $U \subset X$, of a set set $\mathcal{E}(U)$ of functions $U \to E$ such that

(1) If $U \subset V$ are subsets of X and $f \in \mathcal{E}(V)$, then $f|_U \in \mathcal{E}(U)$.
(2) If $V = \bigcup U_i$ is an open cover of V and $f : V \to E$ is a function such that $f|_{U_i} \in \mathcal{E}(U_i)$ for all i, then $f \in \mathcal{E}(V)$.

One can require as the zeroth condition that the empty function $\emptyset \to E$ belongs to $\mathcal{E}(\emptyset)$.
Next, we listed a some examples.

- We can set E to be a topological space and $\mathcal{E}(U)$ to be the set of continuous functions $U \to E$.
- We can take E to be a smooth manifold and $\mathcal{E}(U) = \mathcal{C}^\infty(U)$.
- For k an algebraically closed field, let X be a quasi projective variety over k. Then for $E = k$, we can take $\mathcal{E}(U)$ to be the set of regular functions on U.
- Let X be a smooth curve over k and D a divisor on X. Letting $E = k \cup \{\infty\}$, we can consider $\mathcal{E}(U) = \mathcal{O}(D)_U$.
- Let E be a topological space and $\phi : E \to X$ a continuous function. Then we can consider $\mathcal{E}(U) = \{\sigma \in \mathcal{C}^0(U) : U \to E \mid \phi\sigma = \mathrm{id}\}$.

In addition, we consider the non-example of constant functions on a Hausdorff topological space, say $X = \mathbb{R}$.

Finally, we recalled the definition of a basis for a topological space and pointed out that it is a homework exercise to show that a sheaf of functions defined on a basis can be extended to all the open sets in a topological space.

January 12 – Abstract sheaves. Last time: Defined "concrete" sheaves, i.e. sheaves of E-valued functions for some set E.

These are good enough for many applications, but not quite good enough for us... Namely,

(1) we must deal with nilpotents
(2) we want to deal with e.g. differential forms as objects on our original space X.

Definition. Let X be a topological space. A ***sheaf*** $\mathcal{E}$ on X is the following data:

- for each open $U \subset X$, a set $\mathcal{E}(U)$
- for each pair of nested opens $U \subset V$ a map

$$\rho_U^V \colon \mathcal{E}(V) \to \mathcal{E}(U),$$

such that

(1) for any open $U \subset X$

$$\rho_U^U = \mathrm{id} : \mathcal{E}(U) \to \mathcal{E}(U),$$

and for any nested opens $U \subset V \subset W$

$$\rho_U^W = \rho_U^V \circ \rho_V^W \colon \mathcal{E}(W) \to \mathcal{E}(U).$$

(2) if $\{U_i\}$ is an open cover of V, and we pick $f_i \in \mathcal{E}(U_i)$ that agree on intersections, i.e.
$$\rho^{U_i}_{U_{ij}}(f_i) = \rho^{U_j}_{U_{ij}}(f_j) \text{ where } U_{ij} := U_i \cap U_j,$$
then there exists a unique $g \in \mathcal{E}(V)$ that satisfies $\rho^V_{U_i}(g) = f_i$.

There is now no set E (of "values" of the "functions" inside each $\mathcal{E}(U)$). But it is still good intuition to think of each $\mathcal{E}(U)$ as "sections over U" (so, maps out of U to some set E) and the maps ρ^V_U as "restriction maps".

Condition (1) is the "Presheaf condition" i.e. it says that $\mathcal{E}$ is a contravariant functor from the poset-category $\mathfrak{Top}(X)$ of open subsets of X under inclusions, to the category $\mathfrak{Set}$ of sets.

Condition (2) is the "local gluability" condition, i.e. we can "glue together" all the sections f_i on local patches U_i to get a section g on all of V, and this section g is uniquely determined by the f_i. Some sources (e.g Hartshorne, p. 61) separate the "uniqueness" and "existence" into two separate statements (conditions (3) and (4) in Hartshorne, respectively).

Example. On a smooth manifold

- differential k-forms is a sheaf.
- closed k-forms is a sheaf.
- exact k-forms is <u>not</u> a sheaf. (it is a presheaf, whose sheafification is closed k-forms)

[slight digression]

Question: Is there an example of a presheaf where the uniqueness of gluing fails?

Answer: (dumb example) On any space X, let $\mathcal{E}(X) = \{\pm 1\}$, and for any proper (open) subset $U \subset X$ let $\mathcal{E}(U) = \{1\}$. The restriction maps ρ are all determined by this data, and for any cover $\{U_i\}$ of X by proper open subsets, the local sections $1 \in \mathcal{E}(U_i)$ will glue together to give either $+1$ or $-1 \in \mathcal{E}(X)$ as a global section.

[end digression]

Definition. Let $\mathcal{E}$ be a sheaf on X. The ***stalk*** $\mathcal{E}_x$ of $\mathcal{E}$ at a point $x \in X$ is defined as the direct limit (also known as injective limit or colimit)
$$\mathcal{E}_x = \varinjlim_{U \ni x} \mathcal{E}(U)$$
taken over all opens $U \subset X$ containing x. This means
$$\mathcal{E}_x = \bigsqcup_{U \ni x} \mathcal{E}(U)/\sim,$$
with equivalence relation $\sim$ defined by
$$(f, U) \sim (g, V) \text{ if } \exists W \subset U \cap V \text{ such that } \rho^U_W f = \rho^V_W g \in \mathcal{E}(W).$$

$\star$ Obvious Assertion: the above defines an equivalence relation

$\star$ Obvious Assertion: we get the same stalks if we take the limit <u>only</u> over a basis of opens (e.g. distinguished opens in the Zariski topology)

Definition. A ***germ*** is an equivalence class $(f, U)_x$ in $\mathcal{E}_x$.

Example. On $X = \operatorname{Spec} \mathbb{C}[x, y]/(xy)$, the regular functions $f = 0$ and $g = y$ are equal in the stalk at the point $(1, 0)$, but are not equal in the stalk at $(0, 1)$.

How to turn any sheaf $\mathcal{E}$ back into a concrete sheaf:
Now that we have stalks, let
$$E = \bigsqcup_{x \in X} \mathcal{E}_x \quad .$$
Any $f \in \mathcal{E}(U)$ determines an (honest) function $\sigma(f)\colon U \to E$ sending $x \mapsto (f, U)_x$.
Fact: This $\sigma\mathcal{E}$ is a sheaf of E-valued functions on X, which is isomorphic to $\mathcal{E}$.

Implicit claims here:

- If $f, g \in \mathcal{E}(U)$ and $\sigma(f) = \sigma(g)$, then $f = g$
- If $U \subset V$ and $f \in \mathcal{E}(V)$, then $\sigma(\rho^V_U f) = \sigma(f)|_U$.
- If $\{U_i\}$ covers V and $f_i \in \mathcal{E}(U_i)$ satisfy $\sigma(f_i)|_{U_{ij}} = \sigma(f_j)|_{U_{ij}}$, then there exists $g \in \mathcal{E}(V)$ such that $\sigma(g)|_{U_i} = \sigma(f_i)$.

Example. $X = \operatorname{Spec} \mathbb{C}[x, y]/(xy, y^2) = \operatorname{Spec} A$
The functions 0 and y are equal in the quotient ring $A/\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$, but they are not equal in the localization $A_\mathfrak{p}$ at $\mathfrak{p} = (x, y)$.
(Moral: In a ring with nilpotents, a regular function may not be uniquely determined by its values at all points, but it will be determined uniquely by its values in all stalks.)

January 14 – Sheafification. Today, we first introduce the notion of ***sheafification***. Let X be a topological space, let E be a set, and let $\mathcal{E}$ be a concrete presheaf of E-valued functions on X. For an open set $U \subseteq X$, let $\mathcal{E}^+(U)$ be the collection of functions $f\colon U \to E$ such that for all $x \in U$, there exists an open neighbourhood V of x with $V \subseteq U$ such that $f|_V \in \mathcal{E}(V)$. The sheaf $U \mapsto \mathcal{E}^+(U)$ is the sheafification of the concrete presheaf $\mathcal{E}$.

For a general presheaf $\mathcal{E}$ on a topological space X, recall that the stalk at the point $x \in X$ is the direct limit $\mathcal{E}_x = \varinjlim_{U \ni x} \mathcal{E}(U)$. Define $E = \bigsqcup_{x \in X} \mathcal{E}_x$. For each open set $U \subseteq X$, one may regard $\mathcal{E}(U)$ as a subcollection of functions $U \to E$, so there is a map $\mathcal{E}(U) \to \{\text{functions } U \to E\}$; denote the image by $\mathcal{F}(U)$. Then, $U \mapsto \mathcal{F}(U)$ is a concrete presheaf, and hence we can sheafify as before.

We next discuss ***maps of sheaves***. Let $\mathcal{E}, \mathcal{F}$ be sheaves on a topological space X, then a map of sheaves $\varphi\colon \mathcal{E} \to \mathcal{F}$ is a collection of maps $\varphi_U\colon \mathcal{E}(U) \to \mathcal{F}(U)$, one for each open set $U \subseteq X$, such that the following diagram commutes:
$$\begin{array}{ccc}
\mathcal{E}(U) & \xrightarrow{\rho^U_V} & \mathcal{E}(V) \\
\varphi_U \downarrow & & \downarrow \varphi_V \\
\mathcal{F}(U) & \xrightarrow[\rho^U_V]{} & \mathcal{F}(V)
\end{array}$$
where the ρ^U_V's denote the restriction maps in the appropriate sheaves. Consequently, we can define the ***image*** $\operatorname{Im}(\varphi)$ to be the sheafification of the presheaf $U \mapsto \varphi_U(\mathcal{E}(U))$.

Now, if $\mathcal{E}, \mathcal{F}$ are sheaves of abelian groups and $\varphi\colon \mathcal{E} \to \mathcal{F}$ is a map of sheaves of abelian groups (that is, $\varphi\colon \mathcal{E}(U) \to \mathcal{F}(U)$ is a morphism of abelian groups for every $U \subseteq X$ open), then we can define the ***cokernel*** $\mathcal{C}o\mathcal{K}er(\varphi)$ to be the sheafification of the presheaf $U \mapsto \mathcal{F}(U)/\varphi(\mathcal{E}(U))$. Furthermore, we can define the ***kernel*** $\mathcal{K}er(\varphi)$ to be the presheaf $U \mapsto \ker(\varphi_U)$, which is in fact a sheaf (that is, there is no need to sheafify).

January 16 – Sheaves as an abelian category. At the end of the previous lecture, we defined the ***kernel***, ***cockerel*** and ***image*** of a map of sheaves. Today we stated that sheaves are an abelian category. Informally, this means that you can treat them like modules. Here are some specific ways that is true:

The universal property of the kernel: If we have $\mathfrak{C} \to \mathfrak{A} \xrightarrow{\phi} \mathfrak{B}$ so that $\mathfrak{C} \to \mathfrak{B}$ is 0, then there is a unique map $\mathfrak{C} \to \ker(\phi)$ so that the following diagram

$$\begin{tikzcd} \mathfrak{C} \arrow[r] \arrow[d] & \mathfrak{A} \arrow[r, "\phi"] & \mathfrak{B} \\ \ker(\phi) \arrow[ur] & & \end{tikzcd}$$

is commutative.

The universal property of the cokernel: If $\mathfrak{A} \xrightarrow{\phi} \mathfrak{B} \to \mathfrak{F}$ have $\mathfrak{A} \to \mathfrak{F}$ is 0, there is a unique map $\operatorname{coker}(\phi) \to \mathfrak{F}$ so that the following diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "\phi"] & \mathfrak{B} \arrow[r] \arrow[dr] & \mathfrak{F} \\ & & \operatorname{coker}(\phi) \arrow[u] \end{tikzcd}$$

is commutative.

The key property of images Give a map $\phi : \mathfrak{A} \to \mathfrak{B}$, we have

$$\operatorname{Im}(\phi) \cong \operatorname{Ker}(\mathfrak{B} \to \operatorname{Coker}(\phi)) \cong \operatorname{Coker}(\operatorname{Ker}(\phi) \to \mathfrak{A}).$$

Injectivity: $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is injective $\iff \ker(\phi) = 0 \iff \varphi_U$ is injective for all U.

Surjectivity $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is surjective $\iff \operatorname{coker}(\phi) = 0 \iff$ All maps on stalks are surjective $\iff$ For any $U \subseteq X$ open, any $x \in U$, any $g \in \mathfrak{B}(U)$, there exists V, $x \in V \subseteq U$ and $f \in \mathfrak{A}(V)$ such that $\phi_V f = \rho^U_V g$.

Exactness $\mathfrak{A} \xrightarrow{\alpha} \mathfrak{B} \xrightarrow{\beta} \mathfrak{C}$ is exact $\iff$ The sequence of the stalks is exact. $\iff$ For any $U \subseteq X$ open, any $x \in U$, any $g \in \mathfrak{B}(U)$ such that $\beta(g) = 0$, there exists V, $x \in V \subseteq U$ and $f \in \mathfrak{A}(V)$ such that $g(f) = \rho^U_V g$.

January 21 – Schemes. We introduced the concept of schemes.

Definition. A scheme is a topological space X equipped with a sheaf of commutative rings such that X locally looks like $\operatorname{Spec} A$ for various commutative rings A.

As a set, $\operatorname{Spec} A$ is the set of prime ideals of a commutative ring A. A map of rings $\phi : A \to B$ induces a map $\operatorname{Spec} A \leftarrow \operatorname{Spec} B$ given by $\phi^{-1}(p) \hookleftarrow p$.

Remark. This relies on the obvious claim that the pre image of a prime ideal is prime.

Definition. The Zariski topology on $\operatorname{Spec} A$ is given as follows: For any $S \subseteq A$, set $V(S) = \{p \in \operatorname{Spec} A : S \subseteq p\}$. We say that a set T is closed if $T = V(S)$ for some S.

Distinguished open sets of $\operatorname{Spec} A$ have the form $D(f) = \{p | p \not\ni f\}$. $D(f)$ is homeomorphic to $\operatorname{Spec} f^{-1}A$.

Definition. We define a sheaf called $\mathcal{O}$ on $\operatorname{Spec} A$ given by $\mathcal{O}(D(f)) = f^{-1}A$. We extend this definition to all open sets by gluing.

January 23 – Regular function form a sheaf. The goal of today's class is that a sheaf of rings $\mathcal{O}$ on Spec A is actually a sheaf.

Let A be a commutative ring. What do we need to check? Let $D(f)$ be a distinguished open of Spec A. For $D(f) = \cap D(h_i)$, we are given $c_i \in h_i^{-1}A$ such that $c_i = c_j$ in $(h_ih_j)^{-1}A$. We ultimately want to show that there exists a unique $a \in f^{-1}A$ such that $a = c_i$ for all i in $h_i^1 A$.

More generally, let M be a A- module, and replace every red A with M. Replacing A by $f^{-1}A$ and M by $f^{-1}M$, we can assume that f is a unit, which may as well be 1.

To summarize, we want to show the below theorem:

Theorem. Assume Spec $A = \cup D(h_i)$, given $c_i \in h_i^{-1}A$ such that $c_i = c_j$ in $(h_ih_j)^{-1}A$, there exists unique a in A such that $a = c_i$ in $h_i^{-1}A$.

Proof.

$$\cup D(h_i) = \text{Spec } A \iff \forall i, \text{ no prime contains } h_i$$

The above condition is equivalent to $< h_1^{n_1}, \cdots, h_r^{n_r} >= (1)$ for any $n_1, n_2, \ldots, n_r$.

Uniqueness : Suppose there exists $a, a' \in M$ such that $a = a'$ in h_i^{-1} for all i. Then, $h_i^{n_i}(a - a') = 0$ for some n_i, and because of (a), this implies that $a = a'$: let $\sum b_i h_i^{n_i} = 1$ for some n_i. Then, $\sum b_i h_i^{n_i}(a - a') = 0 \iff 1 \cdot (a - a') = 0 \iff a = a'$.

Now we only need to show the existence.

Existence : Let $c_i = a_i/h_i^{n_i}$ with $c_i \in h_i^{-1}M$, $a_i \in M$ such that $c_i = c_j$ in $(h_ih_j)^{-1}M$ for all i, j. Then, $(h_ih_j)^{m_{ij}}(a_ih_j^{n_j} - a_jh_i^{n_i}) = 0$ for some m_{ij}. Take finitely many $h_1, \cdots, h_r$ such that $\langle h_1, \cdots, h_r\rangle = 1$. The ideal is to pick a large enough N that we can take a“ = ” $\sum b_i h_i^N \frac{a_i}{h_i^{n_i}}$ where $\sum b_i h_i^N = 1$.

Take $N = \max\{n_i\} + \max\{m_{ij}\}$ works. Choose b_i such that $\sum b_i h_i^N = 1$. Set $a = \sum b_i a_i h_i^{N-n_i}$. For any k, we compute that

$$ah_k^{N+n_k} = \sum_i b_i a_i h_i^{N-n_i} h_k^{N+n_k} = \sum_i b_i a_k h_i^N h_k^N = a_k h_k^N.$$

So $a = a_k/h_k^{n_k}$ in $h_k^{-1}A$, as desired. □

January 26 - Morphism of Schemes, start of discussion of Proj. Recall from last time that a scheme is a pair $(X, \mathcal{O})$, where X is a topological space, $\mathcal{O}$ is a sheaf of commutative rings, and $(X, \mathcal{O})$ locally looks like Spec(A) for various commutative rings A. We define a map of scheme as follows:

Definition. A map of schemes: $\phi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is the data such that

- $\phi : X \to Y$ is a continuous map.
- For every $V \subset Y$, there is a map of rings $\phi_V^* : \mathcal{B}(V) \to \mathcal{A}(\phi^{-1}(V))$ such that
  (1) ϕ^* commutes with restriction;
  (2) $\phi^* : \mathcal{B}_{\phi(x)} \to \mathcal{A}_x$ is a local map.

Alternatively, the condition (2) may be replaced by the following:
(2)′ For any affine $V \subset Y$ and affine $U \subset \phi^{-1}(V)$, then map $(U, \mathcal{A}|_U) \to (V, \mathcal{B}|_V)$ is the one induced by the map of rings $\mathcal{B}(V) \to \mathcal{A}(U)$.

One way to get schemes from other schemes is by taking open subsets: If $(X, \mathcal{A})$ is a scheme, and $U \subset X$ is open, then $(U, \mathcal{A})$ is a scheme, and the inclusion map $U \hookrightarrow X$ is a map of schemes.

Next we talked about open sets. If S is a ring, I is an ideal, $V(I) = \{\mathfrak{p} \in \operatorname{Spec} S : \mathfrak{p} \supseteq I\}$, then $\operatorname{Spec} S \backslash V(I)$ is an open set. We pointed out that $\operatorname{Spec} k[x_1, \cdots, x_n] \backslash V(\langle x_1, \cdots, x_n \rangle)$ is a non-affine scheme for $n \geqslant 2$.

Lastly we brought up the notion of graded rings. A $\mathbb{Z}$-graded ring is a ring S equipped with a direct sum decomposition

$$S = \bigoplus_{n=-\infty}^{\infty} S_n, \qquad S_i S_j \subseteq S_{i+j}.$$

Similarly, a $\mathbb{Z}_{\geqslant 0}$-graded ring is a ring satisfying the same conditions with $S = \bigoplus_{n=0}^{\infty} S_n$. The geometric significance of grading over ground field k is that it gives a group action. More explicitly, we said that the grading on a k-algebra S is equivalent to the action $\operatorname{Spec} k[u, u^{-1}]$ of on $\operatorname{Spec} S$.

January 28 - Construction of Proj. For a commutative k-algebra S, a $\mathbb{Z}$ grading on S is equivalent to an action of $G := \operatorname{Spec} k[u, u^{-1}]$ on $\operatorname{Spec} S$. See the problem set for details. The degree 0 part of S is the functions which are invariant for the group action, so we can think of S_0 roughly as function on the quotient.

Suppose that S is $\mathbb{Z}_{\geq 0}$ graded. We define $S_+ = \bigcup_{j>0} S_j$. Intuitively, we want $\operatorname{Proj} S$ to be $S \setminus V(S_+)$, modulo scaling.

In actuality, we build $\operatorname{Proj} S$ from a collection of open patches. For each homogenous $f \in S_+$, one of the patches on $\operatorname{Proj} S$ will be $\operatorname{Spec}(f^{-1}S)_0$. Here $\operatorname{Spec} f^{-1}S$ is a distinguished open in $\operatorname{Spec} S$ and $\operatorname{Spec}(f^{-1}S)_0$ should be thought of as the quotient. Note that $\bigcup_{f \in S_+} D(f) = \operatorname{Spec} S \setminus V(S_+)$.

We could simply define $\operatorname{Proj} S$ by gluing together the patches $\operatorname{Spec}(f^{-1}S)_0$. Specifically, if $f_i \in S_i$ and $f_j \in S_j$, glue $\operatorname{Spec}(f_i^{-1}S)_0$ to $\operatorname{Spec}(f_j^{-1}S)_0$ by gluing the distinguished opens $\operatorname{Spec}(f_j^i/f_i^j)^{-1}(f_i^{-1}S)_0$ and $\operatorname{Spec}(f_i^j/f_j^i)^{-1}(f_j^{-1}S)_0$.

We present another way of describing the same result. We build the underlying topological space of $\operatorname{Proj} S$ directly. The points of $\operatorname{Proj} S$ are homogenous prime ideals in $\operatorname{Spec} S \backslash V(S_+)$, and the topology is inherited from the Zariski topology on $\operatorname{Spec} S$. Implicit in this approach is the claim that the homogenous primes of $f^{-1}S$ are in bijection with the primes of $(f^{-1}S)_0$; see the homework. One then defines a sheaf of regular functions on this set by defining the regular functions on $\{$homogenous primes of $f^{-1}S\}$ to be $(f^{-1}S)_0$.

January 30 - Products over Fields. For today, all schemes are over a field k - we're going to discuss products over k. First, let's consider the affine case.

Let $X = \operatorname{Spec} k[x_1, \ldots, x_m]/\langle f_1, \ldots, f_r \rangle = \operatorname{Spec} A$ and $Y = \operatorname{Spec} k[y_1, \ldots, y_n]/\langle g_1, \ldots, g_s \rangle = \operatorname{Spec} B$. We define the product to be $X \times Y = \operatorname{Spec}[x_1, \ldots, x_m, y_1, \ldots, y_n]/\langle f_1, \ldots, f_r, g_1, \ldots, g_s \rangle$. We might worry about whether or not this definition depends on the choice of generators, but in fact we have $X \times Y = \operatorname{Spec}(A \otimes_k B)$.[2]

For k-algebras A, B, we define $\operatorname{Spec} A \times_{\operatorname{Spec} k} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_k B)$. This construction makes sense for affine schemes, and in general, $X = \bigcup_i \operatorname{Spec} A_i$, $Y = \bigcup_i \operatorname{Spec} B_j$ and we get $X \times_{\operatorname{Spec} k} Y$ by gluing $\operatorname{Spec}(A_i \otimes_k B_j)$. We should, however, worry about whether the gluing conditions are satisfied (this is done in Hartshorne, and amounts to checking formal

[2] Remember that if A, B are R-modules for some commutative ring R, then $A \otimes_R B$ is an R-module on generators $a \otimes b$ with the relations $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$, $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$ and $r(a \otimes b) = ra \otimes b = a \otimes rb$. If A, B are commutative R-algebras, then $A \otimes_R B$ is a commutative ring and $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$.

properties of tensor products), and whether or not this definition depends on the choice of cover.

For the latter, note that we can define the scheme-theoretic product by the usual universal property: if $f : T \to X$, $g : T \to Y$, we want a scheme $X \times_k Y$ and a unique map $\phi : T \to X \times_k Y$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[drr] \arrow[ddr] \arrow[dr, dashed] & & \\
& X \times Y \arrow[r] \arrow[d] & X \\
& Y &
\end{tikzcd}$$

If P_1 and P_2 are two such objects, a standard argument shows that P_1 and P_2 must be isomorphic. For a sketch of existence, note that when X and Y are affine, we obviously have the dual diagram in the category of commutative k-algebras:

$$\begin{tikzcd}
R & & \\
& A \otimes B \arrow[ul, dashed] & A \arrow[l] \arrow[ull] \\
& B \arrow[u] \arrow[uul] &
\end{tikzcd}$$

and we need only patch these together – Hartshorne goes into the details.

The next thing we want to introduce is the notion of a functor of T-points for a fixed scheme T. This is a functor from the category of schemes to the category of sets.

$$X \mapsto \mathrm{Hom}(T, X) =: X(T)$$

$$(X \xrightarrow{f} Y) \mapsto (g \mapsto f \circ g)$$

An example is:

Example. If $k = \overline{k}$ and X is finite type over k, then $(\operatorname{Spec} A)(k) = \mathrm{Hom}_k(A, k) = \operatorname{MaxSpec} A$.

A slightly more interesting example is:

Example. Consider $X = \operatorname{Spec} \mathbb{R}[t]$; then $X(\operatorname{Spec} \mathbb{R}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[t], \mathbb{R}) = \mathbb{R}$. On the other hand, $X(\operatorname{Spec} \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}[t], \mathbb{C}) = \mathbb{C}$.

From the universal property of products, we see that for any T, $X(T) \times Y(T) = (X \times Y)(T)$. If we have a map $G \times G \to G$, we get maps $G(T) \times G(T) \to G(T)$. So an action $G \times X \to X$ gives an action $G(T) \times X(T) \to X(T)$.

A warning: the map on underlying sets from the underlying set of $X \times Y$ to the product of the underlying set of X and the underlying set of Y is not an equality. For instance:

Example. Let $X = Y = \operatorname{Spec} \mathbb{R}[t]$. The closed points of $X \times Y$ are equivalence classes in $\mathbb{C}^2$, where equivalence is given by conjugation. Then $(t^2+1) \in \operatorname{Spec} \mathbb{R}[t]$ and $(u^2+1) \in \operatorname{Spec} \mathbb{R}[u]$, but the ideal $\langle t^2+1, u^2+1\rangle \in \operatorname{Spec}\mathbb{R}[t,u]$ is not prime. The ideals $\langle t-u, t^2+1, u^2+1\rangle$ and $\langle t+u, t^2+1, u^2+1\rangle$ correspond to the sets $\{(i,i),(-i,i)\}$ and $\{(i,-i),(-i,i)\}$ respectively. More generally, for every irreducible polynomial $f(t,u)$, we get a point in $\operatorname{Spec}\mathbb{R}[t,u]$ and we see that we get (in the scheme-theoretic product) uncountably many points lying over a single point in the set-wise product.

February 4 - Assorted conditions on maps. Today we discussed: closed subschemes, finite maps and maps of finite type.

In the affine worlds we want closed subschemes of $\operatorname{Spec} A$ to correspond to ideals of A. For example, if $A = k[x]$, then

$$\operatorname{Spec} k \subset \operatorname{Spec}\frac{A}{(x^2)} \subset \operatorname{Spec} A$$

are closed subschemes.

Definition. If $(X,\mathcal{O})$ is a scheme, a closed subscheme $(Z,\mathcal{A})$ is a scheme with a map $(\phi,\phi^{\#})$ to $(X,\mathcal{O})$ such that ϕ is a closed embedding and $\phi^{\#}$ is surjective.

Since $\phi^{\#}$ is surjective, it gives us an ideal sheaf $\mathcal{I}_Z$ on X, where

$$\mathcal{I}_Z(U) = \operatorname{Ker}(\mathcal{O}(U) \to \mathcal{A}(\phi^{-1}(U))).$$

It is a nontrivial theorem that the closed subschemes of $\operatorname{Spec} A$ correspond exactly to ideals of A: One might imagine that there could be some other closed subschemes which only locally come from ideals without doing so globally. Hartshorne proves this theorem as Exercise II.3.10.(b); a better proof can be found as Corollary II.5.10.

Maps of finite type are the scheme-analogues of maps of rings $A \to B$ inducing a finitely generated A-algebra structure on B. Exercise II.3.1 shows that even if this condition is defined for a particular cover by affine schemes, it still holds for any affine subscheme. Maps of finite type have finite dimensional fibers and can be thought of as "finite dimensional fibers in a really nice way".

Finite maps are the scheme-analogues of maps of rings $A \to B$ which endow B with the structure of a finitely generated A-module (B is module-finite over A). Finite maps have finite fibers. (The converse isn't true though: $Speck[u,u^{-1}] \to \operatorname{Spec} k[u]$ has finite fibers but is not finite.) We spent some time talking about finite maps last term – see October 6, 8 and 10.

February 6 - Separated schemes. A good analogy that one should have for this topic is that *separated* is like *Hausdorff*.

Example. Non-hausdorff manifold Take two disjoint copies of $\mathbb{R}$ and glue $\mathbb{R}^*$ to $\mathbb{R}^*$ by the identity map. The space $X = \mathbb{R}^* \sqcup \mathbb{R}^*/\sim$ is a "line with double point", which is a manifold which is not Hausdorff.

Example. Separated scheme In schemes, take $\operatorname{Spec} k[t]$ and $\operatorname{Spec} k[u]$ and glue the open subsets $D(t)$ and $D(u)$ by $k[t,t^{-1}] \simeq k[u,u^{-1}]$ by $t \mapsto u$. (Note: if we glue the open subsets by $t \mapsto u^{-1}$, we get $\mathbb{P}^1$, which is separated). The space we get, which can be also considered as a "line with double point", is not separated. We will talk more about this example after we give a precise definition.

Definition. Let X be a scheme over S. Then, X is *separated* if the diagonal embedding $X \to X \times_S X$ is a closed embedding.

Note that usually, S is quite simple, such as $\operatorname{Spec} k$ or $\operatorname{Spec} \mathbb{Z}$. To check whether a scheme X is affine, we can actually limit our attention to open covers:

Lemma. Let $X = \cup U_i$. Then, X is separated if and only if all the diagonal maps $U_i \cap U_j \to U_i \times U_j$ are close embeddings

Proof. $U_i \times U_j$'s form an open cover of $X \times X$. □

Now, with the above the definition, let's look at our previous example.

Example.

(1) (*Revisited*) Let X be a line with double points. i.e., $X = (\operatorname{Spec} k[u]) \cup (\operatorname{Spec} k[t])$. For convenience, let $U = \operatorname{Spec} k[u]$, and $V = \operatorname{Spec} k[v]$. We see that $U \cap V \simeq \operatorname{Spec} k[ss^{-1}]$, and the diagonal map is $\triangle : U \cap V \to U \times V$ such that the ring map associated to it sends t, u to s. This is not a closed embedding, because geometrically, $U \times V$ is a rectangle, and $U \cap V$ is a diagonal without one point.

(2) Now we glue U, V as above along $\operatorname{Spec} k[s, s^{-1}]$ with a map

$$U \cap V \to U \times V$$

such that the associated ring map sends $t \mapsto s$, and $u \mapsto s^{-1}$. The,n $k[t, u]$ generates $\operatorname{Spec} k[s, s^{-1}]$, and our diagonal embedding is actually a closed embedding.

Facts

(1) Affine schemes are separated
(2) $\operatorname{Proj} S$ is also separated
(3) Open/closed subschemes of a separated scheme are also separated

The above facts motivate us to give a below definition:

Definition. A scheme is *quasi-projective* if it is an open subscheme of a projective scheme

So basically, one can get non-separated schemes by glueing/ taking quotients.

Places where one should say "separated":

Suppose we have two schemes S and T and two maps $\phi, \psi : S \to T$. Suppose also that there exists a dense subset Z of the underlying set of S such that $\phi|_Z = \psi|_Z$. Then, we want $\phi = \psi$. In order for the condition $\phi|_Z = \psi|_Z$ to imply $\phi = \psi$, we need both S and T to be separated. (See exercise II.4.2.)

If X is separated, and U, V are affine open subschemes, then $U \cap V$ is affine. (See exercise II.4.3. You also might like to look back to Problem 5(a), Problem Set 6, from last term.)

February 9 – Proper schemes. Motivating analogy: <u>proper</u> for schemes is like <u>compact</u> for topological spaces.

As with finite maps, $X \to Y$ proper implies proper fibers, but the converse does not hold. (Take the same counterexample as before, i.e. the hyperbola projected to the affine line)

Definition. A map $\pi : X \to Y$ is ***proper*** if it is separated, of finite type, and universally closed.

Definition. A map $\pi : X \to Y$ is ***universally closed*** if for all $Y' \to Y$, the pullback $X \times_Y Y' \to Y'$ is ***closed***, i.e. any closed subset $Z \subset X \times_Y Y'$ has closed image in Y'.

$$\begin{tikzcd} Z \arrow[r, hook] & X \times_Y Y' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"'] \\ \pi'(Z) \arrow[r, hook] & Y' \arrow[r] & Y \end{tikzcd}$$

Example. The map $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} k$ is not proper.

Take $Y' = \operatorname{Spec} k[y] \to \operatorname{Spec} k$, then the pullback by this map has $\operatorname{Spec} k[x] \times \operatorname{Spec} k[y] = \operatorname{Spec} k[x,y] = \mathbb{A}^2$:

$$\begin{tikzcd} Z = V(xy-1) \arrow[r, hook] & \mathbb{A}^2 \arrow[r] \arrow[d, "\pi'"'] & \mathbb{A}^1 \arrow[d, "\pi"'] \\ \pi'(Z) = \{y \neq 0\} \arrow[r, hook] & \mathbb{A}^1 \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

If we take Z to be the hyperbola cut out by $xy-1$, then the image $\pi'(Z)$ will be $\mathbb{A}^1$ with the origin removed, which is not closed in $\mathbb{A}^1$.

Remark. π' fails to be closed because Z has points "running off to infinity" in the X-fibers.

Example. The map $\mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1] \to \operatorname{Spec} k$ is proper.

We do not prove this, but we observe that the same setup as above does not disprove properness: if we identify $\mathbb{A}^1$ with the affine open $U_0 = \{[x_0 : x_1] | x_0 \neq 0\} = \{[1 : x]\}$, then the equation $xy - 1 = (x_1/x_0)y - 1$ homogenizes to $x_1 y - x_0$:

$$\begin{tikzcd} Z = V(x_1 y - x_0) \arrow[r, hook] & \mathbb{P}^1 \times \mathbb{A}^1 \arrow[r] \arrow[d, "\pi'"'] & \mathbb{P}^1 \arrow[d, "\pi"'] \\ \pi'(Z) = \mathbb{A}^1 \arrow[r, hook] & \mathbb{A}^1 \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Now the image $\pi'(Z)$ is all of $\mathbb{A}^1$, since the point "at infinity" of $\mathbb{P}^1$ i.e. $[x_0 : x_1] = [0 : 1]$ is in the fiber above the origin $y = 0$.

Nice properties of properness

(1) Properness is local on the target Y, i.e. if $U \subset Y$ and $\pi \colon X \to Y$ is proper, then $\pi^{-1}(U) \to U$ is proper, and conversely if $\{U_i\}$ is an open cover of Y and each $\pi^{-1}(U_i) \to U_i$ is proper, then $X \xrightarrow{\pi} Y$ is proper.
(2) Properness is stable under base change, i.e. if $X \xrightarrow{\pi} Y$ is proper and $U \to Y$ is any map, then $X \times_Y U \to U$ is proper. (This generalizes the first half of (1) where $U \hookrightarrow Y$ is an inclusion)
(3) If $X \xrightarrow{\pi} Z$ is proper and factors through $X \xrightarrow{f} Y \xrightarrow{g} Z$, i.e. $\pi = g \circ f$, then $X \xrightarrow{f} Y$ is proper. ($Y \xrightarrow{g} Z$ does not have to be proper)
(4) Images of proper maps are closed. (universally closed implies closed)
(5) If S is a $\mathbb{Z}_{\geq 0}$-graded ring, finitely generated over S_0, then $\operatorname{Proj} S \to \operatorname{Spec} S_0$ is proper.
(6) Finite maps are proper. (follows from Nakayama's lemma)
(7) If Y is noetherian and $X \xrightarrow{\pi} Y$ has finite fibers (π is ***quasi-finite***), then the following are equivalent:

- π is finite
- π is projecive
- π is proper.

Added by David: I tried to sketch a proof of (5) at the end of class and did a poor job of it. Let's try again. Our claim is local on Y', so we may take $Y' = \operatorname{Spec} T_0$. Let T be the graded ring $S \otimes_{S_0} T_0$. Then $X \times_Y Y' = \operatorname{Proj} T$ and we just need to prove that the map $\operatorname{Proj} T \to \operatorname{Spec} T_0$ is closed. Let Z be a closed subset of $\operatorname{Proj} T$, corresponding to a graded ideal $I = \bigoplus_j I_j$ in T. Then $\mathfrak{p} \in \operatorname{Spec} T_0$ is in the image of Z if and only if $\operatorname{Proj} T/\mathfrak{p}T$ is nonempty. Now, $\operatorname{Proj} T/\mathfrak{p}T$ is nonempty if and only if $(T/\mathfrak{p}T)_j \neq 0$ for all sufficiently large j. Since an intersection of closed sets is closed, we simplest must show that the set of $\mathfrak{p}$ for which $(T/\mathfrak{p}T)_j$ is nonzero is closed. Since T is a finitely generated graded T_0 algebra, T_j is a finitely generated T_0-module. Write T_j as the cokernel of $\phi : T_0^M \to T_0^N$, where we might have to take M infinite if S_0 is not noetherian, but N is finite. So we can think of ϕ as an $M \times N$ matrix. We want to study the set of primes $\mathfrak{p}$ for which the matrix $\phi \bmod \mathfrak{p}$ is not of full rank. This set is Zariski closed, since it is cut out by the $N \times N$ minors of ϕ.

February 11 – The valuative criterion. Let $X \to \operatorname{Spec} k$ be a proper map, let V be a k-scheme, and let $U \subset V$ be an open dense subset. Let $\phi : U \to X$. We have the open inclusion of U into V, and we get maps $U \xrightarrow{\psi} X \times V$ that can project down to $\operatorname{Spec} k$ through either X or V (as in the diagram below). Properness says $\overline{\psi(U)}$ projects to a closed subset of V, so there are points of $\overline{\psi(U)}$ over all of V.

$$\begin{tikzcd} & & U \arrow[dl, "\psi"'] \\ X \arrow[d] & X \times V \arrow[l] \arrow[d] & \\ \operatorname{Spec} k & V \arrow[l] & \end{tikzcd}$$

The valuative criterion asserts that we can extend $\phi\colon U \to X$ to a morphism $\overline{\phi}\colon V \to X$.

For example, suppose $V = \operatorname{Spec} k[[t]]$, and $U = \operatorname{Spec} k((t))$. If $X = \mathbb{P}^2 = \operatorname{Proj} k[x, y, z]$, then we can map $U \to \operatorname{Spec} k[\frac{x}{z}, \frac{y}{z}] \subset \mathbb{P}^2_k$ by e.g. $\frac{x}{z} \mapsto t^{-1} + 3t + \ldots$ and $\frac{y}{z} \mapsto 7t^{-1} + \ldots$. The valuative criterion says that a solution in power series in t, t^{-1} can be extended to a solution in power series in t.

In general, such an extension does not exist: consider the map $\mathbb{P}^2 \setminus \{(0:0:1)\} \to \mathbb{P}^1$ given by $(x:y:z) \mapsto (x:y)$. This does not extend to all of $\mathbb{P}^2$, because local rings on surfaces are not necessarily valuation rings.

Let us show the valuative criterion in the case $R = k[[t]]$ and $K = \operatorname{Frac} R = k((t))$. Consider:

$$\begin{tikzcd} & & \operatorname{Spec} K \arrow[dl, "\psi"'] \\ X \arrow[d, "\text{proper}"'] & X \times_{\operatorname{Spec} R} \operatorname{Spec} R \arrow[l] \arrow[d] & \\ \operatorname{Spec} k & \operatorname{Spec} R \arrow[l] & \end{tikzcd}$$

We can factor ψ as $\operatorname{Spec} K \to \overline{\psi(\operatorname{Spec} K)} \to X \times_{\operatorname{Spec} R} \operatorname{Spec} R$. There is a point $q \in X \times \operatorname{Spec} R$ over any $t_0 \in \operatorname{Spec} R$. Let $p \in X$ be the image of q in X, and pass to an open affine $\operatorname{Spec} A$ containing p. This allows us to draw the corresponding diagram on rings:

$$\begin{tikzcd}
& & & K \\
& & B \arrow[ur, hook] & \\
A \arrow[r] \arrow[urr, bend left, "\alpha^*"] \arrow[urrr, bend left, "\phi^*"] & A[[t]] \arrow[ur] & & \\
& k[[t]] \arrow[u] & &
\end{tikzcd}$$

We need to show that $\phi^*(A) \subset k[[t]]$. Suppose not: say $\phi^*(a) = t^{-n} \cdot (\text{stuff})$, then $\alpha^*(a) = t^{-n} \cdot (\text{stuff})$. It follows that $B \otimes_R R/tR = B \otimes_R k = 0$, so there are no points of $\overline{\psi(\operatorname{Spec} K)}$ above t_0, a contradiction. This completes the proof.

February 13 – Examples of gluing coherent sheaves. We discussed some examples of gluing coherent sheaves.

Example 1: char $k \neq 2$, $A = k[x,y]/(y^2 - x^3 + x)$, $\Omega^1_A = Adx + Ady/(2ydy - (3x^2-1)dx)$. We have two charts $D(y)$ and $D(3x^2-1)$. We can talk about the elements $\frac{dx}{2y} \in y^{-1}\Omega^1_A$ and $\frac{dy}{3x^2-1} \in (3x^2-1)^{-1}\Omega^1_A$. On the overlap $D(y(3x^2-1))$, we have $\frac{dx}{2y} = \frac{dy}{3x^2-1}$. So the sheaf property says that there should be some ω in Ω^1_A which restricts to $\frac{dx}{2y}$ on $D(y)$ and $\frac{dy}{3x^2-1}$ on $D(3x^2-1)$.

We can find ω explicitly. The condition that $\operatorname{Spec} A = D(y) \cup D(3x^2-1)$ means that y and $3x^2-1$ must generate the unit ideal. Explicitly $-\frac{9}{2}y^2x + (3x^2-1)(\frac{3}{2}x^2-1) = 1$. So

$$\omega = (-\frac{9}{2}y^2x + (3x^2-1)(\frac{3}{2}x^2-1))\omega = (-\frac{9}{2}xy)\frac{dx}{2} + (\frac{3}{2}x^2-1)dy.$$

But we can also think about ω without having to explicitly find a formula for it.

We started **Example 2:** Take $A = k[x,y]/(y^2 - x^3 + x)$, $B = k[u,v]/(v^2 + u^3 - u)$. Glue $\operatorname{Spec} A$ to $\operatorname{Spec} B$. $D(x) \simeq D(u)$. $x = \frac{1}{u}$, $y = \frac{v}{u^2}$, take Ω^1_A glue to Ω^1_B. We have $x^{-1}\Omega^1_A \simeq u^{-1}\Omega^1_B$. Glue $\frac{dx}{2y} = \frac{du}{2v}$ to get a sheaf Ω^1. This shows how we can talk about 1-forms as sections of a sheaf on non-affine spaces. A good exercise is to work out that the global sections of this Ω^1 are one dimensional. More generally, Problem 4 on Problem Set 10, from Fall Term, works out the case of global 1-forms on a hyper elliptic curve of genus g and shows that there is a g-dimensional space of them.

We decided to retreat to **Example 1.5:** Line bundles on $\mathbb{P}^1$. Let $U = \operatorname{Spec} k[u]$, $V = \operatorname{Spec} k[v]$. Glue $D(u)$ to $D(v)$, u to v^{-1}. This gives $\mathbb{P}^1$. Let's build a locally free sheaf $\mathcal{C}$ of rank 1 on $\mathbb{P}^1$.

Let $\mathcal{C}$ on U be $\widetilde{k[u]\alpha}$ and let $\mathcal{C}$ on V be $\widetilde{k[u]\beta}$. Then

$$\begin{aligned}
(\widetilde{k[u]\alpha})(D(u)) &= u^{-1}k[u]\alpha = k[u,u^{-1}]\alpha \\
(\widetilde{k[u]\beta})(D(v)) &= v^{-1}k[v]\beta = k[v,v^{-1}]\beta
\end{aligned}$$

We can glue β to $u^m\alpha$ for any $m \in \mathbb{Z}$. Then

$$\mathcal{C}(\mathbb{P}^1) = \{(f,g) | f \in \mathcal{C}(U), g \in \mathcal{C}(V) \ : \ f|_{U\cap V} = g|_{U\cap V}\}.$$

Putting $f = p(u)\alpha$ and $g = q(v)\beta$ for polynomials p and q, we have $p(u)\alpha = q(u^{-1})u^m\alpha$. So p can be any polynomial of degree $\leq m$, and q is its reversal. There is an $m+1$ dimensional space of global sections. This is the line bundle $\mathcal{O}(m)$.

February 16 – Examples of the Proj **construction.** We discussed further examples of gluing coherent sheaves.

Example 1: Recall the computation from the end of last time. Write $\mathbb{P}^1 = U \cup V$ where $U = \operatorname{Spec} k[u]$ and $V = \operatorname{Spec} k[v]$. Define a coherent sheaf $\mathcal{C}$ on $\mathbb{P}^1$ by $\mathcal{C}(U) = k[u] \cdot \alpha$, $\mathcal{C}(V) = k[v] \cdot \beta$ (the free modules on the generators α, β, respectively). Glue on $U \cap V$ by $\beta = u^m\alpha$.
This gives the global sections $\mathcal{C}(\mathbb{P}^1) = \{(f,g), f \in \mathcal{C}(U), g \in \mathcal{C}(V) : f|_{U\cap V} = g|_{U\cap V}\}$. Given a global section (f,g), write $f = p(u)\alpha, g = q(v)\beta$ for polynomials $p \in k[u], q \in k[v]$. We need to have $p(u)\alpha = q(v)\beta$. Since the gluing on $\mathbb{P}^1$ is $v = u^{-1}$, $p(u)\alpha = q(u^{-1}) \cdot u^m\alpha$. The global sections $\Gamma(\mathcal{C})$ are generated by $\{\alpha, u\alpha, \cdots, u^m\alpha\} = \{v^m\beta, v^{m-1}\beta, \cdots, v\beta, \beta\}$.

Example 2: The same example, from a different perspective. Write $\mathbb{P}^1 = \operatorname{Proj} k[x,y]$ with the cover $U = \operatorname{Spec}(y^{-1}k[x,y])_0 = k[\frac{x}{y}]$, $V = \operatorname{Spec}(x^{-1}k[x,y])_0 = \operatorname{Spec} k[\frac{y}{x}]$ and let $u = \frac{x}{y}, v = \frac{y}{x}$.
The overlap is $\operatorname{Spec}((xy)^{-1}k[x,y])_0 = \operatorname{Spec} k[\frac{x}{y}, \frac{y}{x}]$. The coherent sheaf $\mathcal{O}(m)$ (see Hartshorne) is defined by by $\mathcal{O}(m)(U) = (y^{-1}S)_m = k[\frac{x}{y}] \cdot y^m$ and $\mathcal{O}(m)(V) = (x^{-1}S)_m = k[\frac{y}{x}] \cdot y^m$. Inside $((xy)^{-1}S)_m$, we have $x^m = (\frac{x}{y})^m y^m$. In other words, this is the previous example with $\beta = x, u = \frac{x}{y}, \alpha = y^m$.

We now discuss some examples that show the subtleties of turning graded modules into sheaves.

Example 3: Let $S = k[x,y]/xy$ with the usual grading. $\operatorname{Spec} S$ is the union of the x- and y-axes. So $\operatorname{Spec} S \setminus V(x,y)$ is the axes excluding the origin. We claim $\operatorname{Proj} S$ is simply two points.

Let's work through this from the definitions. We obtain $\operatorname{Proj} S$ by gluing $\operatorname{Spec}(x^{-1}S)_0$ and $\operatorname{Spec}(y^{-1}S)_0$. Now, $x^{-1}S = x^{-1}k[x,y]/xy = x^{-1}k[x,y]/y = x^{-1}k[x]$ so $(x^{-1}S)_0 = k$. The corresponding chart of $\operatorname{Proj} S$ is $\operatorname{Spec} k$. Similarly, the other chart is $\operatorname{Spec} k$. We have $(xy)^{-1}S = \{0\}$. So the overlap is $\operatorname{Spec}\{0\} = \emptyset$ and $\operatorname{Proj} S$ is two points.

We have $\mathcal{O}(\operatorname{Proj} S) = k^2$. Notice that S_0 is only k. Let's see how this happens from the definitions. We have $\mathcal{O}(\operatorname{Proj} S) = \operatorname{Ker}\Big((x^{-1}S)_0 \oplus (y^{-1}S)_0 \to ((xy)^{-1}S)_0\Big)$. The map sends $(f,g) \mapsto f|_{U\cap V} - g|_{U\cap V}$, so its kernel is all of $x^{-1}S)_0 \oplus (y^{-1}S)_0 = k \oplus k$.

Example 4: Take $S = k[u^4, u^3v, uv^3, v^4] \subset k[u,v]$ (note that u^2v^2 is **not** in S). Our grading is such that $S_1 = k \cdot \{u^4, u^3v, uv^3, v^4\}$ (so degree-4 terms in the usual grading become degree-1). We can also think of this as $S = k[p,q,r,s]/\langle p^2r - q^3, qs^2 - r^3, ps - qr\rangle$.

We have $\operatorname{Proj} S = \operatorname{Spec}(p^{-1}S)_0 \cup \operatorname{Spec}(s^{-1}S)_0$. We have $(p^{-1}S)_0 = k[\frac{v}{u}, \frac{v^3}{u^3}, \frac{v^4}{u^4}] = k[\frac{v}{u}]$ and $(s^{-1}S)_0 = k[\frac{u}{v}]$. So $\operatorname{Proj} S$ is simply $\mathbb{P}^1$.

Let's build the sheaf corresponding to the degree 1 part of S; we'll call it $\mathcal{C}$. We have $\mathcal{C}(U) = k[\frac{v}{u}] \cdot u^4$ and $\mathcal{C}(V) = k[\frac{u}{v}] \cdot v^4$. Gluing gives $\mathcal{C}(\mathbb{P}^1) = k[u^4, u^3v, u^2v^2, uv^3, v^4]$ even though $S_1 = k \cdot \{u^4, u^3v, uv^3, v^4\}$.

Finally, we stated but did not prove the following facts:

Let S be a noetherian commutative ring. Then the abelian categories of quasi-coherent (respectively, coherent) sheaves on $\operatorname{Spec} S$ and S-modules (resp., finitely generated S-modules) are equivalent. The equivalence sends a module M to the sheaf $\widetilde{M}$, and sends the sheaf $\mathcal{E}$ to the module $\Gamma(\mathcal{E})$.

In the projective world, given S a $\mathbb{Z}_{\geq 0}$-graded ring, the categories of quasi-coherent (resp., coherent) sheaves on $\operatorname{Proj} S$ and (finitely generated) graded S-modules *up to low degree* are equivalent. We send the module M to $\tilde{M}$; we send the sheaf $\mathcal{E}$ to $\bigoplus_{j\geq 0}\Gamma(\mathcal{E}(j))$.

February 18 – Vector Bundles and Locally Free Sheaves. In today's class, we talked about vector bundles and locally free sheaves. We began with the following definition:

Definition. Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-module, then $\mathcal{E}$ is said to be **locally free** of rank r if there is a open cover $\{U_i\}$ of X such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ for all i. In particular, $\mathcal{E}$ is a coherent sheaf.

Then we introduced the notion of vector bundle.

Definition. A rank r vector bundle is a scheme $\mathcal{E}$ over X (namely, $\pi : E \to X$) with maps

$$\alpha : E \times_X E \longrightarrow E, \mathcal{M} : \mathbb{A}^1_X \times_X E \longrightarrow E,$$

which locally look like addition and scalar multiplication in the sense that for any $x \in X$ we have the following maps

$$\alpha : \mathbb{A}^r_U \times_U \mathbb{A}^r_U \longrightarrow \mathbb{A}^r_U, \qquad \text{(addition)} \mathcal{M} : \mathbb{A}^1_U \times_U \mathbb{A}^r_U \longrightarrow \mathbb{A}^r_U, \qquad \text{(scalar multiplication)}$$

for some open subset U of $x \in X$.

Then we described the correspondence between vector bundle E of rank r and locally free sheaf $\mathcal{E}$ of rank r by the following constructions: (See Exercise II 5.18 for details) **(1).** Given the vector bundle $E \xrightarrow{\pi} X$ of rank r, we define a sheaf on X by:

$$\mathcal{E}(U) = \{\text{sections of } U \to \pi^{-1}(U)\}$$

It is a sheaf with abelian group structure. Also it is an $\mathcal{O}_X$-module: Given $f \in \mathcal{O}_X(U)$, $U \to \pi^{-1}(U)$, f gives a map $\phi : U \to \mathbb{A}^1_U$, then we define

$$\phi \times \sigma : U \longrightarrow \mathbb{A}^1_U \times_U \pi^{-1}(U) \xrightarrow{\mathcal{M}} \pi^{-1}(U).$$

(2). Conversely, given a locally free sheaf $\mathcal{E}$, we want to construct a vector bundle E corresponding to it. There are two routes to do so:

(i) (messy) Take an open cover U_i of X on which we have isomorphisms $\phi_i : \mathcal{O}_{U_i}^{\oplus r} \cong \mathcal{E}|_{U_i}$. Then we have maps $g_{ij} : \mathcal{O}_{U_i\cap U_j}^{\oplus r} \to \mathcal{O}_{U_i\cap U_j}^{\oplus r}$ given by

$$\mathcal{O}_{U_i\cap U_j}^{\oplus r} \xrightarrow{\phi_i} \mathcal{E}_{U_i\cap U_j} \xrightarrow{\phi_j} \mathcal{O}_{U_i\cap U_j}^{\oplus r}.$$

On triple overlap $U_i \cap U_j \cap U_k$, we have

$$g_{ij}|_{U_i\cap U_j\cap U_k} \circ g_{jk}|_{U_i\cap U_j\cap U_k} = g_{ik}|_{U_i\cap U_j\cap U_k}.$$

Each g_{ij} is an $r\times r$ matrix with entries in $\mathcal{O}(U_i\cap U_j)$, so it gives an automorphism of $\mathbb{A}|^r_{U_i\cap U_j}$. Use these to glue the $\mathbb{A}^r_{U_i}$ together.

(ii) (slicker) We may just consider the case where $X = \operatorname{Spec} R$ as it glues to a scheme. Let M be a locally free R-module of rank r. Take $E = \operatorname{Spec}\operatorname{Sym}\cdot M^\vee$, where $M^\vee = \operatorname{Hom}(M, R)$ and

$$\operatorname{Sym}^\bullet M^\vee = R \oplus M^\vee \oplus \operatorname{Sym}^2 M^\vee \oplus \cdots$$

in which $\mathrm{Sym}^2 M^\vee = M^\vee \otimes_R M^\vee/(m_1 \otimes m_2 - m_2 \otimes m_1)$ and so forth. It is clear that $\mathrm{Sym}^\bullet M^\vee$ is a commutative ring. If M is locally free, then the sections $\mathrm{Spec}\, R \to E$ corresponds to M, as the ring map $\mathrm{Sym}^\bullet M^\vee \to R$ is determined by the map $M^\vee \to R$ and $(M^\vee)^\vee \cong M$.

February 20 – Line bundles and divisors. A line bundle is a locally free sheaf of rank 1. Cartier divisors correspond to line bundles most generally, while Weil divisors are more geometric. From now, assume we're on an integral, Noetherian, locally factorial scheme. (Locally factorial means that local rings are UFD's. Just as a reminder, there are plenty of non-UFD local rings in the world: Localize $k[x, y]/(y^2 - x^3)$ at $\langle x, y\rangle$ or localize $k[x, y, z]/(xz - y^2)$ at $\langle x, y, z\rangle$.) These aren't the weakest possible hypotheses, but we're not aiming for the weakest possible hypotheses.

Local rings at the height 1 primes are discrete valuation rings; so for any height 1 prime $\mathfrak{p}$, and any $f \in \mathrm{Frac}\, \mathcal{O}_\mathfrak{p}$, we can talk about $v_\mathfrak{p}(f)$.

A *Weil divisor* on X is a formal sum of height 1 primes (possibly with negative coefficients), and we write

$$\mathrm{Div}(X) = \mathbb{Z} \cdot \{\text{height 1 primes}\}.$$

Since X is integral, it has a generic point η with fraction field $K = \mathrm{Frac}\, \mathcal{O}_\mathfrak{p}$ for every $\mathfrak{p} \in X$. For $f \in K^*$, the divisor is

$$\mathrm{Div}(f) = \sum_{\mathrm{ht}(\mathfrak{p})=1} v_\mathfrak{p}(f) \cdot [\mathfrak{p}].$$

Such a divisor is called principal, and the class group $C\ell(X)$ is

$$C\ell(X) = \mathrm{Div}(X)/\text{principal divisors}.$$

In this setup, we have:

Theorem. With these hypotheses on X, the group of line bundles on X is isomorphic to $C\ell(X)$.

Let's sketch how this works. In one direction, given a divisor $D = \sum d_\mathfrak{p} \cdot [\mathfrak{p}]$ we define a sheaf $\mathcal{O}(D)(U) = \{f \in K \mid v_\mathfrak{p}(f) + d_\mathfrak{p} \geq 0 \text{ for all } \mathfrak{p} \in U\}$

Given a line bundle $\mathcal{L}$ on X, the stalk $\mathcal{L}_\eta$ is a one dimensional K vector space. (Here η is the generic point and $K = \mathrm{Frac}\, \mathcal{O}_\eta$ is the common value of every $\mathrm{Frac}\, \mathcal{O}_\mathfrak{p}$. At any height 1 prime $\mathfrak{p}$, choose a generator $\tau_\mathfrak{p}$ for $\mathcal{L}_\mathfrak{p}$ as an $\mathcal{O}_\mathfrak{p}$-module. So $\sigma_\mathfrak{p} \in \mathcal{L}_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} k$, and $\sigma_\mathfrak{p} = f_\mathfrak{p} \tau_\mathfrak{p}$. The corresponding divisor is going to be $D = \sum v_\mathfrak{p}(f_\mathfrak{p}) \cdot [\mathfrak{p}]$.

As an example, let's take $\mathbb{P}^1 = U \cup V = \mathrm{Spec}\, k[u] \cup \mathrm{Spec}\, k[v]$ glued by $u = v^{-1}$. Then $\mathcal{O}(m)\big|_U = k[u]\alpha$ and $\mathcal{O}(m)\big|_V = k[v]\beta$. A section here is something like $c_0\alpha + c_1 u\alpha + c_2 u^2 \alpha + \dots c_m u^m \alpha = c_0 v^m \beta + \dots c_m \beta$. This vanishes at the roots of $c_m x^m + \dots + c_1 x y^{m-1} + c_0 y^m$.

Finally, note that we can pull back line bundles. If $X \xrightarrow{\phi} Y$ and $L \xrightarrow{\pi} Y$ is a line bundle, then $X \times_Y L \to X$ is a line bundle. The corresponding operation on coherent sheaves is taking $\phi^*\mathcal{L}$:

$$(\phi^*\mathcal{L})(U) = (\mathcal{O}(V)) \otimes_{\varinjlim \mathcal{O}(U)} (\varinjlim_{V \supset \phi(U)} \mathcal{L}(V))$$

February 23 – Line bundles and maps to $\mathbb{P}^n$. From a Projective space to a line bundle: Given $X \xrightarrow{\phi} \mathbb{P}^{n-1}$, we get $\phi^*(\mathcal{O}(1))$, a line bundle on X.

$$\begin{tikzcd} X \times_{\mathbb{P}^{n-1}} \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ X \arrow[r, "\phi"] & \mathbb{P}^{n-1} \end{tikzcd}$$

Every section of $\mathcal{L} \to \mathbb{P}^{n-1}$ pulls back to a section of $X \times_{\mathbb{P}^{n-1}} \mathcal{L}$, namely to a section of $\phi^*(\mathcal{O}(1))$.

Remark. This is behind the group law on elliptic curves. Given an elliptic curve $L \subseteq \mathbb{P}^2$ and two lines L_1, L_2, $L_1 \cap L = \{P_1, Q_1, R_1\}$, $L_2 \cap L = \{P_2, Q_2, R_2\}$. Then $\mathcal{O}(P_1+Q_1+R_1) \cong \mathcal{O}(P_2+Q_2+R_2) \cong \phi^*(\mathcal{O}(1))$ where $\phi : L \hookrightarrow \mathbb{P}^2$.

From line bundles to projective space: Let $\mathcal{L}$ be a line bundle on X, let $\sigma_1, ..., \sigma_n$ be a basis of $\Gamma(X, \mathcal{L})$, let $B \subseteq X$ be the closed subscheme $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 0$.

Definition. B is called the base points of $\mathcal{L}$.

Thus we get a map $X \backslash B \to \mathbb{P}^{n-1}, x \mapsto (\sigma_1(x), \cdots \sigma_n(x))$.

Let U_i be the open set in X where σ_i is nonzero, thus, $\mathcal{L}|_{U_i} = \mathcal{O}|_{U_i} \cdot \sigma_i$ and on U_i, $\sigma_j = f_{j/i}\sigma_i$ for some $f_{j/i} \in \mathcal{O}(U_i)$, $(f_{1/i}, \cdots, f_{n/i})$ gives a map $U_i \to \mathbb{A}^{n-1}$. These glue to a map $X \to \mathbb{P}^{n-1}$.

More generally, given $W \subseteq \Gamma(\mathcal{L})$, we get a map $X \backslash B \to \mathbb{P}(W^*)$.

Definition. A linear series on X is a line bundle $\mathcal{L}$ and $W \subseteq \Gamma(\mathcal{L})$ as in the paragraph above.

Definition. If X is proper, then $\mathcal{L}$ on X is called very ample if $\mathcal{L}$ has no base points and $X \to \mathbb{P}(\Gamma(\mathcal{L})^*)$ is a closed immersion. $\mathcal{L}$ is called ample if $\mathcal{L}^{\otimes n}$ is very ample for some $n \in \mathbb{Z}_+$.

Added after class: I thought about the issue of non-proper varieties more, and I agree with Hartshorne that it is good, for an arbitrary X, to say that a line bundle $\mathcal{L}$ is very ample if $\mathcal{L} \cong \phi^*\mathcal{O}(1)$ for some immersion $\phi : X \to \mathbb{P}^n$. I also agree with Takumi (see `http://math.stackexchange.com/questions/85688`) that Hartshorne has a bad definition of immersion. An immersion should be defined as the composition of **first** a closed immersion and **then** an open immersion. EGA does this; Hartshorne puts them in the other order. These should be the same with enough Noetherian hypotheses, but in any case where they differ, I think EGA's choice is better.

February 25 – More on line bundles and maps to projective space. Let X be a scheme over a field k. Today we discussed how to obtain a map $X \to \mathbf{P}^n$ from (sections of) a line bundle $\mathcal{L}$ on X. The construction is as follows. First, we pick a vector subspace W of $\Gamma(X, \mathcal{L})$ that we call a *linear system*. Concretely, on closed points we can choose a basis $\sigma_1, \ldots, \sigma_n$ of W, and then define the map

$$X \setminus B \to \mathbf{P}(W^*) \cong \mathbf{P}^{n-1}$$
$$x \mapsto [\sigma_1(x) : \sigma_2(x) : \cdots : \sigma_n(x)]$$

where B is the subset of X on which the σ_i simultaneously vanish.

Here is a first example:

Example (Cuspidal cubic)**.** Let $X = \mathbf{A}^1_k$ and $\mathcal{L} = \mathcal{O}_X$. Then, $\Gamma(\mathcal{L}) \cong k[t]$ is infinite dimensional over k. Choosing $W = k\{1, t^2, t^3\}$, i.e., the k-vector subspace of $k[t]$ spanned by $1, t^2.t^3$, we get a map

$$\begin{aligned} \mathbf{A}^1 &\to \mathbf{P}^2 \\ t &\mapsto [1 : t^2 : t^3] \end{aligned}$$

whose image in $\mathbf{P}^2$ is the cuspidal cubic minus the point at ∞. Note that if we were allowed to talk about (very) amplitude of linear systems, this linear system would not be very ample because the map it induces is not an immersion; in particular, the local sections at the origin do not induce a surjection onto regular functions on $\mathbf{A}^1$.

Instead, if we take the linear system spanned by t^2, t^3, then $B = \{0\}$ are our base points. We therefore get a map

$$\begin{aligned} \mathbf{A}^1 \setminus \{0\} &\to \mathbf{P}^1 \\ t &\mapsto [t^2 : t^3] = [1 : t] \end{aligned}$$

whose image is $\mathbf{P}^1 \setminus \{0, \infty\}$.

In the same vein, we have the following example:

Example ($\mathbf{P}^1$ as a quotient space)**.** Let $X = \mathbf{A}^2_k$, and $\mathcal{L} = \mathcal{O}_X$. Then, $\Gamma(X, \mathcal{L}) = k[x, y]$. If we take W to be the vector subspace spanned by x, y, then $B = \{0\}$, and we get a map

$$\begin{aligned} \mathbf{A}^2 \setminus \{0\} &\to \mathbf{P}^1 \\ (x, y) &\mapsto [x : y] \end{aligned}$$

which we recall is the quotient map used to define $\mathbf{P}^1$ last semester.

Here is a more involved example:

Let X be the hyperelliptic curve obtained by glueing the two curves $\{v^2 = u + u^{2g+1}\}$ and $\{y^2 = x^{2g+1} + x\}$ in $\mathbf{A}^2$ by $(u, v) = (x^{-1}, yx^{-(2g+1)})$.

Let ∞ be the unique point in $\{u = v = 0\}$. In the local ring at ∞, we have $u = v^2(\text{unit}) = v^2(1 + u^{2g})$, hence $v\mathcal{O}_\infty$ generates $\mathfrak{m}_\infty$. The local ring $\mathcal{O}_\infty$ is a DVR. We compute the some valuations of elements in $\mathcal{O}_\infty$:

$$\nu_\infty(v) = 1, \quad \nu_\infty(u) = 2, \quad \nu_\infty(x) = -2, \quad \nu_\infty(y) = 1 + (g + 1)(-2) = -(2g + 1)$$

Note that $g = 1$ gives an elliptic curve.

Now $\Gamma(X, \mathcal{O}_X) = k$ (since for any complete projective curve, there is no nonconstant polynomial that doesn't blow up at ∞). We investigate what happens if we allow poles at ∞. First, note

$$\frac{k[x, y]}{y^2 = x^{2g+1} + x} = k[x] + k[x]y \text{ (as a } k[x]\text{-module)}$$

So, any polynomial $k[x, y]$ can be expressed as $f(x) + h(x)y$. We then can calculate

$$\begin{aligned} \nu_\infty(f(x) + g(x)y) &= \min\{\nu_\infty(f(x)), \nu_\infty(h(x))\nu_\infty(y)\} \\ &= \min\{-2 \deg f, -(2 \deg h + 2g + 1)\} \end{aligned}$$

Thus, even if we allow a pole at ∞, we gain nothing: $\Gamma(X, \mathcal{O}_X(\infty)) = k$.

On the other hand, $\Gamma(X, \mathcal{O}_X(2\infty)) = k + k \cdot x$. Note that $\mathcal{O}(2\infty)_\infty = \mathcal{O}_\infty \cdot x$. The section $x \in \Gamma(X, \mathcal{O}_X(2\infty))$ is nonvanishing at ∞, while the section $1 \in \Gamma(X, \mathcal{O}_X(2\infty))$ is vanishing at ∞, since $1 = u \cdot x$ and $u \in \mathfrak{m} \subset \mathcal{O}_\infty$. Thus, the linear system spanned by $1, x$ has no base points, inducing the map

$$\begin{aligned} X &\xrightarrow{\varphi} \mathbf{P}^1 \\ (x,y) &\mapsto [1 : x] \\ (u,v) &\mapsto [u : 1] \end{aligned}$$

which is a twofold cover, where we note the maps glue since $u = x^{-1}$, and $\mathcal{O}(2\infty) = \varphi^*\mathcal{O}(1)$.

We can allow even higher order poles at ∞:

$$\Gamma(X, \mathcal{O}(3\infty)) = \begin{cases} k + kx + ky & \text{if } g = 1 \\ k + kx & \text{if } g > 1 \end{cases}$$

The case $g = 1$ gives an embedding

$$\begin{aligned} E &\hookrightarrow \mathbf{P}^2 \\ (x,y) &\mapsto [1 : x : y] \\ (u,v) &\mapsto \left[\frac{1}{y} : \frac{x}{y} : 1\right] \end{aligned}$$

so choosing a point in E induces an embedding of E as a cubic curve in $\mathbf{P}^2$. With some work, we can show this is a closed embedding, so $\mathcal{O}(3P)$ where P is a point in E is very ample on E a genus 1 curve.

If $g > 1$, then ∞ is a base point of $\mathcal{O}(3\infty)$. This is because the section 1 has a pole of order 0 at ∞ and x has a pole of order 2 at ∞, and so both sections have poles of order less than 3, and therefore have vanishing at ∞. Thus, ∞ is a base point for our linear system, and we get a map $X \setminus \{\infty\} \to \mathbf{P}^1$.

In general, a line bundle of the form $\mathcal{O}(n\infty)$ has a base point for n odd, and has no base points for n even, until $n = 2g + 1$ when y becomes a section. Before this point, the induced map is a double cover of higher and higher degree rational curves, and when $n = 2g + 1$ we get an honest embedding of our curve in $\mathbf{P}^n$. Thus, $\mathcal{O}(\infty)$ is an ample divisor on a hyperelliptic curve.

Finally, $\mathcal{O}((2g-2)\infty)$ is the canonical class of a genus g hyperelliptic curve; the discussion above can be translated into the language of differential forms.

February 27 – Various remarks about line bundles. Why are we thinking about line bundles? Because we sometimes don't have enough actual functions (e.g. on connected projective varieties, we only have constant functions). As a concrete example, a cubic in $\mathbb{P}^2$ is a section of a line bundle, but not the zero locus of a polynomial. A *very ample* line bundle is one that embeds into $\mathbb{P}^n$.

Some comments on ampleness: If X is proper (smooth?) and $\mathcal{L}$ is a line bundle on X, then for every curve C in X, we have a line bundle $\mathcal{L}\big|_C$ which has a degree. If $\mathcal{L}$ is ample on X, then $\mathcal{L}\big|_C$ is certainly ample on C. In that case, $\deg \mathcal{L}\big|_C > 0$. In fact, $\mathcal{L}$ is ample if and only if $\deg \mathcal{L}\big|_C > 0$ is ample for every curve $C \subset X$. There isn't really an analogue of this for very ample line bundles, and is maybe the first place where ample starts looking better.

A discussion of Proj **and the twisting sheaf**: Throughout, let S be a positively graded ring which is generated in degree 1. Remember that $\operatorname{Proj} S$ is glued from $\operatorname{Spec}(f^{-1}S)_0$.

Why are we doing this, and what does this look like? $\operatorname{Spec} f^{-1}S$ is open in $\operatorname{Spec} S$, and the union is $\bigcup_f \operatorname{Spec} f^{-1}S = \operatorname{Spec} S \setminus V(\{f \mid f \in S_{>0}\})$. So we get a map

$$\begin{aligned} \operatorname{Spec} S \setminus V(\{f \mid f \in S_{>0}\}) &\to \operatorname{Proj} S \\ \operatorname{Spec} f^{-1}S &\mapsto \operatorname{Spec}(f^{-1}S)_0 \end{aligned}$$

which can be thought of as 'quotienting by dilation'.

Given a graded module $M = \bigoplus M_j$ for S, the sheaf $\widetilde{M}$ on $\operatorname{Proj} S$ is $\widetilde{M}(D_+(f)) = (f^{-1}M)_0$, which is clearly an $(f^{-1}S)_0$-module. On the module side, we get a twist by $M[n]_i = M_{i+n}$. In particular, we get a line bundle $\mathcal{O}(n) := \widetilde{S[n]}$. Then $\widetilde{M[n]} \cong \widetilde{M} \otimes_{\mathcal{O}} \mathcal{O}(n)$.

Let's look on $\mathbb{P}^1$. There are two charts, $\mathbb{P}^1 = U \cup V$, where $U = \operatorname{Spec} k[x]$ and $V = \operatorname{Spec} k[x^{-1}]$. Then $\mathcal{O}(n)(U) = k[x] \cdot \alpha$ and $\mathcal{O}_n(V) = k[x] \cdot \beta$, and we glue these together by $\beta = x^n \alpha$.

If $E \to \mathbb{P}^1$ is a vector bundle, with could have (say) $E\big|_U \cong \mathbb{A}^r \times U$ and $E\big|_V \cong \mathbb{A}^r \times V$, glued by $g \in \mathrm{GL}_r(k[x, x^{-1}])$. Twisting replaces g for E by $x^n g$ for $E \otimes_{\mathcal{O}} \mathcal{O}(n)$. If $\mathcal{E}(U)$ has basis $e_1, \dots, e_n$ and $\mathcal{E}(V)$ has basis $f_1, \dots, f_n$, then $(\mathcal{E} \otimes \mathcal{O}(n))(U)$ has basis $\alpha \otimes e_1, \dots, \alpha \otimes e_n$ and $(\mathcal{E} \otimes \mathcal{O}(n))(V)$ has basis $\beta \otimes e_1, \dots, \beta \otimes e_n$.

Some functors between categories we've seen so far:

$$\begin{tikzcd}
\text{Graded } k[x_0,\dots,x_r]\text{-modules} \arrow[dr, "M\mapsto\widetilde{M}"] \arrow[dd, bend left, "\text{saturate}"] & \\
& \text{Quasi-coherent sheaves on } \mathbb{P}^r \arrow[dl, "\mathcal{E}\mapsto\bigoplus_{j\geq 0}\Gamma(\mathcal{E}(j))"] \\
\text{Saturated graded } k[x_0,\dots,x_r]\text{-modules} \arrow[uu, bend left, "\text{embed}"] &
\end{tikzcd}$$

The downward map, saturation, is the composition of the two diagonal arrows. When I is a graded ideal of $k[x_0, \dots, x_r]$, the saturation of I is $\bigcup_j (I : \mathfrak{m}^\infty)$, where $\mathfrak{m} = \langle x_0, \dots, x_r \rangle$. See Section 15.4 in Vakil for a good discussion.

March 9 – Introduction to sheaf cohomology. Let X be a topological space, and let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be three sheaves of abelian groups on X. Then, recall that if the sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

is exact, then the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{A}) \longrightarrow \Gamma(X, \mathcal{B}) \longrightarrow \Gamma(X, \mathcal{C})$$

is exact, but we can't extend this to a short exact sequence. We have the following examples:

Example. Let X be a smooth manifold. Let $\underline{\mathbb{R}}$ be the sheaf of locally constant $\mathbb{R}$-valued functions, $\mathcal{C}^\infty$ the sheaf of smooth functions, and Z^1 the sheaf of smooth closed 1-forms. Then, we have the short exact sequence

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{C}^\infty \longrightarrow Z^1 \longrightarrow 0$$

by the Poincaré lemma. We then have the sequence

$$0 \longrightarrow H^0(X, \mathbb{R}) \longrightarrow \mathcal{C}^\infty(X) \xrightarrow{d} Z^1(X) \longrightarrow H^1_{\mathrm{DR}}(X, \mathbb{R})$$

which is exact. $H^1_{\mathrm{DR}}(X, \mathbb{R})$ "measures the failure of integrability of 1-forms on X."

Example. Let X be a complex manifold. Let $\underline{\mathbb{Z}}$ be the sheaf of locally constant $\mathbb{Z}$-valued functions, $\mathcal{H}$ the sheaf of holomorphic functions, and $\mathcal{H}^*$ the sheaf of nonvanishing holomorphic functions. We have the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{H} \xrightarrow{\exp} \mathcal{H}^* \longrightarrow 0$$

which gives the exact sequence

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow \mathcal{H}(X) \longrightarrow \mathcal{H}^*(X) \longrightarrow H^1(X, \mathbb{Z}).$$

These two examples suggest that there might be something called $H^1(X, \mathcal{F})$ for $\mathcal{F}$ a sheaf that extends these exact sequences further to the right.

Example. We also have the following algebro-geometric example. Let X be a smooth projective curve over an algebraically closed field k. Let D be a divisor on X, and P a point in X. From last semester, we have the short exact sequence

$$0 \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}(D+P) \longrightarrow k_P \longrightarrow 0$$

where k_P denotes the skyscraper sheaf at P. This gave the five-term exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{O}(D)) \longrightarrow \Gamma(X, \mathcal{O}(D+P)) \longrightarrow k \longrightarrow H^1(X, \mathcal{O}(D)) \longrightarrow H^1(X, \mathcal{O}(D+P)) \longrightarrow 0$$

We want to mimic these examples for arbitrary sheaves of abelian groups. More precisely, we want to construct functors

$$\begin{aligned} H^q(X, -) \colon \{\text{sheaves of abelian groups on } X\} &\to \{\text{abelian groups}\} \\ \mathcal{E} &\mapsto H^q(X, \mathcal{E}) \end{aligned}$$

satisfying the following three properties:

(1) $H^0(X, \mathcal{E}) = \Gamma(X, \mathcal{E}) = \mathcal{E}(X)$,
(2) every short exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ gives a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^q(X, \mathcal{A}) \arrow[r] & H^q(X, \mathcal{B}) \arrow[r] & H^q(X, \mathcal{C}) \arrow[dll, "\delta"] \\ & H^{q+1}(X, \mathcal{A}) \arrow[r] & H^{q+1}(X, \mathcal{B}) \arrow[r] & H^{q+1}(X, \mathcal{C}) \arrow[r] & \cdots \end{tikzcd}$$

(3) The map δ above satisfies a naturality property (see Hartshorne, III.1).

Definition. Suppose the family of functors $H^q(X, -)$ exists. A sheaf of abelian groups $\mathcal{I}$ on X is ***acyclic*** if $H^i(X, \mathcal{I}) = 0$ for all $i > 0$.

We show that such a family of functors will be uniquely determined by the data of which objects are acyclic, assuming that there are enough maps to acyclic objects.

Proposition. If $0 \to \mathcal{A} \to \mathcal{I} \to \mathcal{B} \to 0$ is exact and $\mathcal{I}$ is acyclic, then

$$0 \longrightarrow H^0(X, \mathcal{A}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{B}) \longrightarrow H^1(X, \mathcal{A}) \longrightarrow 0$$

is exact and $H^q(X, \mathcal{A}) \cong H^{q-1}(X, \mathcal{B})$ for $q \geq 2$.

Proof. Follows by the long exact sequence in property (2). $\square$

Definition. A long exact sequence of sheaves of abelian groups on X of the form

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots$$

where the $\mathcal{I}^q$ are acyclic is called an ***acyclic resolution*** of $\mathcal{A}$. We often use the shorthand $0 \to \mathcal{A} \to \mathcal{I}^\bullet$ to denote an acyclic resolution.

We assume for now a technical lemma about the category of sheaves on X:

Lemma. For any sheaf $\mathcal{A}$ of abelian groups on X, there exists an acyclic sheaf $\mathcal{I}$ such that $\mathcal{A} \hookrightarrow \mathcal{I}$ is injective.

We can now state our theorem for the day:

Theorem. Let $\mathcal{A}$ be a sheaf of abelian groups on X. Then, an acyclic resolution $0 \to \mathcal{A} \to \mathcal{I}^\bullet$ exists, and $H^q(X, -)$ is uniquely determined (up to isomorphism) on objects to be

$$H^q(X, \mathcal{A}) \cong \frac{\ker(H^0(X, \mathcal{I}^q) \to H^0(X, \mathcal{I}^{q+1}))}{\operatorname{Im}(H^0(X, \mathcal{I}^{q-1}) \to H^0(X, \mathcal{I}^q))}.$$

Remark. Note that this theorem does not say anything about what $H^q(X, -)$ should do on morphisms; this will later be a consequence of property (3).

Proof. Define $\mathcal{A}^0 := \mathcal{A}$. By the Lemma, there exists an acyclic sheaf $\mathcal{I}^0$ such that $\mathcal{A}^0 \hookrightarrow \mathcal{I}^0$. We now proceed inductively. Suppose $\mathcal{A}^q, \mathcal{I}^q$ exist. Let $\mathcal{A}^{q+1} := \operatorname{cok}(\mathcal{A}^q \to \mathcal{I}^q)$. By the Lemma again, there exists an acyclic sheaf $\mathcal{I}^{q+1}$ such that $\mathcal{A}^{q+1} \hookrightarrow \mathcal{I}^{q+1}$. This gives short exact sequences

$$0 \longrightarrow \mathcal{A}^q \longrightarrow \mathcal{I}^q \longrightarrow \mathcal{A}^{q+1} \longrightarrow 0$$

for every q, hence by the Proposition, we have the isomorphisms

$$H^q(X, \mathcal{A}^0) \cong H^{q-1}(X, \mathcal{A}^1) \cong \cdots \cong H^{q-i}(X, \mathcal{A}^i)$$

for all $i > 0$, and for $i = 0$, the Proposition gives

$$H^1(X, \mathcal{A}^{q-1}) \cong \operatorname{cok}(H^0(X, \mathcal{I}^{q-1}) \to H^0(X, \mathcal{A}^q))$$

$$H^0(X, \mathcal{A}^q) \cong \ker(H^0(X, \mathcal{I}^q) \to H^0(X, \mathcal{A}^{q+1})) = \ker(H^0(X, \mathcal{I}^q) \to H^0(X, \mathcal{I}^{q+1}))$$

where the last equality is by the fact that $H^0(X, \mathcal{A}^{q+1}) \hookrightarrow H^0(X, \mathcal{I}^{q+1})$. Combining these facts, we have

$$\begin{aligned} H^q(X, \mathcal{A}^0) &\cong \operatorname{cok}(H^0(X, \mathcal{I}^{q-1}) \to \ker(H^0(X, \mathcal{I}^q) \to H^0(X, \mathcal{I}^{q+1}))) \\ &= \frac{\ker(H^0(X, \mathcal{I}^q) \to H^0(X, \mathcal{I}^{q+1}))}{\operatorname{Im}(H^0(X, \mathcal{I}^{q-1}) \to H^0(X, \mathcal{I}^q))}. \end{aligned}$$

Note that the short exact sequences used above can be chained together to form a long exact sequence as in the diagram

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
 & & & \mathcal{A}^1 \arrow[ur] \arrow[dr] & & & & \mathcal{A}^3 \arrow[ur] \arrow[dr] & & & \\
 & & \mathcal{I}^0 \arrow[ur] \arrow[rr] & & \mathcal{I}^1 \arrow[rr] \arrow[dr] & & \mathcal{I}^2 \arrow[rr] \arrow[ur] & & \mathcal{I}^3 \arrow[rr] \arrow[dr] & & \cdots \\
 & \mathcal{A}^0 \arrow[ur] & & & & \mathcal{A}^2 \arrow[ur] \arrow[dr] & & & & \mathcal{A}^3 \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

since by definition $\mathcal{A}^q = \mathrm{Im}(\mathcal{I}^{q-1} \to \mathcal{I}^q)$, and so the maps $0 \to \mathcal{A}^0 \to \mathcal{I}^0$ plus the maps in the middle row give an acyclic resolution $0 \to \mathcal{A} \to \mathcal{I}^\bullet$. Hence we have shown that we can compute the cohomology of $\mathcal{A}^0$ as the cohomology of a complex of abelian groups. □

Example. On a smooth manifold of dimension n,

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathcal{C}^\infty \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \cdots \longrightarrow \Omega^n \longrightarrow 0$$

is an acyclic resolution of $\underline{\mathbb{R}}$. Thus, the Theorem shows $H^q_{\mathrm{DR}}(X, \mathbb{R}) \cong H^q(X, \underline{\mathbb{R}})$.

March 11 – Defining sheaf cohomology via injective resolutions. (Note: in this section, all sheaves are sheaves of abelian groups)

Recall **Leray's theorem** from last time: if $0 \to \mathcal{A} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ is an acyclic resolution of $\mathcal{A}$, then necessarily

$$\begin{aligned} H^q(\mathcal{A}) &\cong \frac{\mathrm{Ker}(\Gamma(\mathcal{I}^q) \to \Gamma(\mathcal{I}^{q+1}))}{\mathrm{Im}(\Gamma(\mathcal{I}^{q-1}) \to \Gamma(\mathcal{I}^q))} \\ &= H^q(0 \to \Gamma(\mathcal{I}^0) \to \Gamma(\mathcal{I}^1) \to \cdots). \end{aligned}$$

We now address the issue of deciding which sheaves are acyclic.

Definition. A sheaf $\mathcal{I}$ is ***injective*** if for any map $f\colon \mathcal{A} \to \mathcal{I}$ and any injection $i\colon \mathcal{A} \hookrightarrow \mathcal{B}$, we may factor f through i:

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "i"] \arrow[d, "f"] & \mathcal{B} \arrow[dl, dashed] \\ \mathcal{I} \end{tikzcd}$$

(i.e. we may "extend" $\mathcal{A} \to \mathcal{I}$ to some $\mathcal{B} \to \mathcal{I}$).

For comparison, in the category of abelian groups $\mathbb{Q}/\mathbb{Z}$ is an injective object.

Theorem. For any sheaf $\mathcal{A}$, there is an injection $\mathcal{A} \hookrightarrow \mathcal{I}$ with $\mathcal{I}$ injective.
(The same result holds for $\mathcal{O}_X$-modules and quasi-coherent $\mathcal{O}_X$-modules.)

This is often expressed by saying there are 'enough injectives' in the category of sheaves of abelian groups. As a corollary, we get:

Corollary. Any sheaf $\mathcal{A}$ has an injective resolution

$$0 \to \mathcal{A} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots.$$

To prove this, we simply find an injection $\mathcal{A} \to \mathcal{I}^0$, and then find $\mathcal{A}^1 \to \mathcal{I}^1$ where $\mathcal{A}^1 = \mathcal{C}o\mathcal{K}er(\mathcal{A} \to \mathcal{I}^0)$, and so on. With this, we can define sheaf cohomology:

Definition (Sheaf cohomology)**.** For any sheaf $\mathcal{A}$ and any injective resolution $0 \to \mathcal{A} \to \mathcal{I}^\bullet$, we define the ***sheaf cohomology*** of $\mathcal{A}$ to be

$$H^q(\mathcal{A}) = H^q(\Gamma(\mathcal{I}^\bullet)).$$

(In particular, this definition implies that injective sheaves are acyclic. Some reasoning for why we would want this to happen is given below)

There is a notion dual to injective that is often more intuitive:

Definition. An object P is ***projective*** if, for any map $f\colon P \to A$ and any surjection $q\colon B \twoheadrightarrow A$, we may factor f through q:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "f"] \\ B \arrow[r, two heads, "q"] & A \end{tikzcd}$$

(i.e. we may "lift" $P \to A$ to some $P \to B$).

Remark. Projective is 'like' free. In particular, free objects are always projective (or, in most categories?), and in the category of modules over a commutative ring, the projective modules are exactly the direct summands of free modules.

<u>A hint that $H^q(\mathcal{I})$ wants to be 0 for $q \geq 1$</u>: given a short exact sequence

$$0 \to \mathcal{I} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0$$

with $\mathcal{I}$ injective, we can show that $0 \to \Gamma(\mathcal{I}) \to \Gamma(\mathcal{B}) \to \Gamma(\mathcal{C}) \to 0$ is exact. To see this, look at:

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\alpha"] \arrow[d, equal] & \mathcal{B} \arrow[dl, dashed, "\pi"] \\ \mathcal{I} & \end{tikzcd}$$

So by injectivity $\mathcal{I} \to \mathcal{B}$ splits $\mathcal{B} = \mathrm{Im}(\alpha) \oplus \mathrm{Ker}(\pi)$, and β induces $\mathrm{Ker}(\pi) \xrightarrow{\sim} \mathcal{C}$. Our sequence $0 \to \mathcal{I} \to \mathrm{Im}(\alpha) \oplus \mathrm{Ker}(\pi) \to \mathcal{C} \to 0$ decomposes into two (very short) exact sequences

$$0 \to \mathcal{I} \to \mathrm{Im}(\alpha) \to 0 \quad \text{and} \quad 0 \to \mathrm{Ker}(\pi) \to \mathcal{C} \to 0,$$

and hence

$$0 \to \Gamma(\mathcal{I}) \to \Gamma(\alpha(\mathcal{I})) \oplus \Gamma(\mathrm{Ker}(\pi)) \to \Gamma(\mathcal{C}) \to 0$$

is exact.

We still need to do the following:

- figure out what H^q is on morphisms $\mathcal{A} \xrightarrow{f} \mathcal{B}$,
- check $H^q(\mathcal{A})$ doesn't depend on our choice of $\mathcal{I}^\bullet$,
- check $H^q(\mathcal{A} \xrightarrow{f} \mathcal{B})$ doesn't depend on choices $\mathcal{A} \to \mathcal{I}^\bullet$, $\mathcal{B} \to \mathcal{J}^\bullet$,
- define the boundary map δ, and
- check the long exact sequence.

Until we do this, we'll write $H^q(\mathcal{A}, \mathcal{I}^\bullet)$.

Lemma. Suppose we have a morphism $\mathcal{A} \xrightarrow{f} \mathcal{B}$ and injective resolutions $\mathcal{A} \to \mathcal{I}^\bullet$ and $\mathcal{B} \to \mathcal{J}^\bullet$. Then we can extend this to a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "f"] & \mathcal{I}^0 \arrow[r] \arrow[d, dashed, "f^0"] & \mathcal{I}^1 \arrow[r] \arrow[d, dashed, "f^1"] & \cdots \\ \mathcal{B} \arrow[r] & \mathcal{J}^0 \arrow[r] & \mathcal{J}^1 \arrow[r] & \cdots \end{tikzcd}$$

Proof. $\mathcal{A} \to \mathcal{I}^0$ is an injection and we have a map $\mathcal{A} \to \mathcal{J}^0$. So we can extend by injectivity: Then $\mathcal{I}^0/\mathcal{A} \to \mathcal{I}^1$ is injective and f_0 induces a map $\mathcal{I}^0/\mathcal{A} \to \mathcal{J}^1$, so we can again extend to $\mathcal{I}^1 \xrightarrow{f^1} \mathcal{J}^1$ by injectivity. We then repeat this argument to finish the proof. $\square$

With notation as above:

Definition. $H^q(f) : H^q(\mathcal{A}, \mathcal{I}^\bullet) \to H^q(\mathcal{B}, \mathcal{J}^\bullet)$ is the map $H^q(f) : H^q(\Gamma(\mathcal{I}^\bullet)) \to H^q(\Gamma(\mathcal{J}^\bullet))$.

We continue in our quest to check that things make sense with:

Lemma. If we have injective resolutions:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "f"] & \mathcal{I}^0 \arrow[r] \arrow[d, bend right, "g^0"'] \arrow[d, bend left, "h^0"] & \mathcal{I}^1 \arrow[r] \arrow[d, bend right, "g^1"'] \arrow[d, bend left, "h^1"] & \cdots \\
\mathcal{B} \arrow[r] & \mathcal{J}^0 \arrow[r] & \mathcal{J}^1 \arrow[r] & \cdots
\end{tikzcd}$$

where the complex with vertical maps $g^\bullet$ commutes, and the complex with vertical maps $h^\bullet$ commutes, then there are maps $\int : \mathcal{I}^{q+1} \to \mathcal{J}^q$ such that $g^j - h^j = d\int - \int d$ for each j.

$$\begin{tikzcd}
 & \mathcal{I}^j \arrow[r, "d"] \arrow[dl, "\int"'] \arrow[d, bend right, "g^j"'] \arrow[d, bend left, "h^j"] & \mathcal{I}^{j+1} \arrow[dl, "\int"] \\
\mathcal{J}^{j-1} \arrow[r, "d"] & \mathcal{J}^j &
\end{tikzcd}$$

This lemma gives us a 'chain homotopy': which implies that $g^\bullet$ and $h^\bullet$ induce the same maps on cohomology by general homological algebra: given $\phi \in H^q(\Gamma(\mathcal{I}^\bullet))$, let $\tilde{\phi} \in \Gamma(\mathcal{I}^q)$ be some representative. Then $g^q(\tilde{\phi}) - h^q(\tilde{\phi}) = \int d\tilde{\phi} - d\int\tilde{\phi} = -d(\int\tilde{\phi})$ since $\tilde{\phi} \in \operatorname{Ker}(d)$, and clearly $-d(\int\tilde{\phi}) \equiv 0$ in $H^q(\Gamma(\mathcal{I}^\bullet))$. Thus $g - h$ induces the zero map on cohomology, so we have $g = h$ on cohomology as claimed.

At this point, we know that $H^q(f) : H^q(\mathcal{A}, \mathcal{I}^\bullet) \to H^q(\mathcal{B}, \mathcal{J}^\bullet)$ is well defined. To see $H^q(\mathcal{A})$ is well-defined (independent of $\mathcal{I}^\bullet$), apply this with $f = \operatorname{Id} : \mathcal{A} \to \mathcal{A}$. This gives $H^q(\mathcal{A}, \mathcal{I}^\bullet) \to H^q(\mathcal{A}, \mathcal{J}^\bullet)$. We also have a map the other way. Then

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{I}^0 \arrow[r] \arrow[d] & \cdots \\
\mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{J}^0 \arrow[r] \arrow[d] & \cdots \\
\mathcal{A} \arrow[r] & \mathcal{I}^0 \arrow[r] & \ldots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{I}^0 \arrow[r] \arrow[d, equal] & \cdots \\
\mathcal{A} \arrow[r] & \mathcal{I}^0 \arrow[r] & \ldots
\end{tikzcd}$$

induce the same map $H^q(\mathcal{A}, \mathcal{I}^\bullet) \to H^q(\mathcal{A}, \mathcal{I}^\bullet)$. So $H^q(\mathcal{A}, \mathcal{I}^\bullet) \to H^q(\mathcal{A}, \mathcal{J}^\bullet)$ and $H^q(\mathcal{A}, \mathcal{J}^\bullet) \to H^q(\mathcal{A}, \mathcal{I}^\bullet)$ are inverse, so in particular they are both isomorphisms.

March 13 – Construction of hypercohomology. Today we'll talk about hypercohomology; this is an interlude, which should clarify some sheaf cohomology questions and help with what's coming up. However, it will remain a side topic for us.

We'll write $\mathfrak{Comp}$ for the category of complexes of sheaves. That is, the objects are

$$0 \to \mathcal{A}^0 \to \mathcal{A}^1 \to \ldots$$

and the morphisms are given by vertical maps which make the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}^0 \arrow[r] \arrow[d] & \mathcal{A}^1 \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & \mathcal{B}^0 \arrow[r] & \mathcal{B}^1 \arrow[r] & \dots \end{tikzcd}$$

commute.

We'll write $\mathcal{H}^q(\mathcal{A}^\bullet) = \mathfrak{Ker}(\mathcal{A}^q \to \mathcal{A}^{q+1})/\mathfrak{Im}(\mathcal{A}^{q-1} \to \mathcal{A}^q)$ (Curly words$\Rightarrow$ Sheaves) Note that $\mathcal{A}^\bullet$ is exact if and only if $\mathcal{H}^\bullet(\mathcal{A}^\bullet) = 0$. We have the notion of a *quasi-isomorphism* of complexes:

Definition. A map $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a *quasi-isomorphism* if f induces isomorphisms $\mathcal{H}^\bullet(\mathcal{A}^\bullet) \xrightarrow{f^*} \mathcal{H}^\bullet(\mathcal{B}^\bullet)$.

Here's an important example of a quasi-isomorphism:

Example. Let $\mathcal{A}$ be a sheaf, and let $\mathcal{A}[0]$ be the complex $0 \to \mathcal{A} \to 0 \to 0 \to \dots$. If $0 \to \mathcal{A} \to \mathcal{I}^\bullet$ is an injective resolution, then $\mathcal{A}[0] \to \mathcal{I}^\bullet$ is a quasi-isomorphism: that is, the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] & \mathcal{I}^2 \arrow[r] & \dots \end{tikzcd}$$

commutes.

A lemma we glossed over:

Lemma. Any complex $\mathcal{A}^\bullet$ has a quasi-isomorphism to a complex of injective objects.

Now, define hypercohomology to be a functor $\mathfrak{Comp} \to \mathbf{Ab}$ satisfying:

- If $f^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a quasi-isomorphism, then $\mathbb{H}^q(f^\bullet)$ is an isomorphism.
- If $\mathcal{I}^\bullet$ is an injective complex, then $\mathbb{H}^q(\mathcal{I}^\bullet) = H^q(\Gamma(\mathcal{I}^\bullet))$.

For any complex $\mathcal{A}^\bullet$, if $\mathcal{A}^\bullet \to \mathcal{I}^\bullet$ is a quasi-isomorphism, then $\mathbb{H}^q(\mathcal{A}^\bullet) = H^q(\Gamma(\mathcal{I}^\bullet))$.

Lemma. (No harder than before) Let $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ be injective. If f and g are quasi-isomorphisms below, we get a quasi-isomorphism h in the following diagram:

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[r, "f"] \arrow[d, "g"] & \mathcal{I}^\bullet \arrow[dl, dashed, "h"] \\ \mathcal{J}^\bullet & \end{tikzcd}$$

Lemma. If $\mathcal{I}^\bullet \xrightarrow{f} \mathcal{J}^\bullet$ is a quasi-isomorphism, then $H^q(\Gamma(\mathcal{I}^\bullet)) \cong H^q(\Gamma(\mathcal{J}^\bullet))$.

Using the first of these two, we can fill in the diagram:

$$\begin{tikzcd} \mathcal{A}^\bullet \arrow[r, "g"] \arrow[d, "f"] & \mathcal{I}^\bullet \arrow[d, dashed] \\ \mathcal{B}^\bullet \arrow[r, "h"] & \mathcal{J}^\bullet \end{tikzcd}$$

if g and h are quasi-isomorphisms.

Lemma. If we have $\mathcal{I}^\bullet \underset{g^\bullet}{\overset{f^\bullet}{\longrightarrow}} \mathcal{J}^\bullet$ and $f^\bullet$, $g^\bullet$ induce the same map on $\mathcal{H}^\bullet$, then $f^\bullet$ and $g^\bullet$ are chain homotopic.

Now, suppose we have $0 \to \mathcal{A}^\bullet \to \mathcal{B}^\bullet \to \mathcal{C}^\bullet \to 0$ exact. That is, for all q, $0 \to A^q \to B^q \to C^q \to 0$ is exact. Then consider:

$$\begin{tikzcd}
\mathcal{A}^0 \arrow[r, "d"] \arrow[dr, "\alpha^0"] & \mathcal{A}^1 \arrow[r, "d"] \arrow[dr, "\alpha^1"] & \mathcal{A}^2 \arrow[r, "d"] \arrow[dr, "\alpha^2"] & \dots \\
& \mathcal{B}^0 \arrow[r, "d"] & \mathcal{B}^1 \arrow[r, "d"] & \dots
\end{tikzcd}$$

Define $\mathcal{K}^\bullet$ by $\mathcal{K}^i = \mathcal{A}^i \oplus \mathcal{B}^{i-1}$. We have a map $\mathcal{K}^\bullet \to \mathcal{C}^{\bullet-1}$ by projecting $\mathcal{K}^q$ onto $\mathcal{B}^{q-1}$ then using the map to $\mathcal{C}^{q-1}$ from the short exact sequence, which is a quasi-isomorphism. So $\mathbb{H}^q(\mathcal{K}^\bullet) \cong \mathbb{H}^{q-1}(\mathcal{C}^\bullet)$ and

$$\begin{tikzcd}
\mathbb{H}^q(\mathcal{A}^\bullet) \arrow[r] & \mathbb{H}^q(\mathcal{B}) \arrow[r] \arrow[dr] & \mathbb{H}^q(\mathcal{C}) \arrow[d, "\cong"] & \\
& & \mathbb{H}^{q+1}(\mathcal{K}^\bullet) \arrow[r] & \mathbb{H}^{q+1}(\mathcal{A}^\bullet)
\end{tikzcd}$$

is functoriality.

To see how this relates to geometry, we have the below theorem by Grothendieck:

Theorem (Grothendieck[3])**.** If X is a smooth finite type scheme over $\mathbb{C}$, then

$$H^q(X^{an}, \mathbb{C}) \cong \mathbb{H}^q(\mathcal{O} \to \Omega^1 \to \Omega^2 \to \dots)$$

If X is affine, $H^q(X, \mathbb{C}) = H^q(\Gamma(\Omega^\bullet))$. If X is projective, we have:

Theorem (Hodge)**.** $H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega^p)$

Example. On a curve, the DeRham complex $\mathfrak{DR}^\bullet$ is $\mathcal{O} \to \Omega^1$. We have a short exact sequence of complexes $0 \to \Omega^1[1] \to \mathfrak{DR}^\bullet \to \mathcal{O}[0] \to 0$. That is:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 \arrow[d] \\
\mathcal{O} \arrow[r, "d"] \arrow[d] & \Omega^1 \\
\mathcal{O} \arrow[r] & 0
\end{tikzcd}$$

[3] *On the de Rham Cohomology of Algebraic Varieties*, Inst. Hautes Études Sci. Publ. Math. No. 29 1966 95 –103

So we have a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & H^0(X^{an},\mathbb{C}) \arrow[r] & \Gamma(\mathcal{O}) \arrow[dll, rounded corners] \\ & \Gamma(\Omega^1) \arrow[r] & H^1(X^{an},\mathbb{C}) \arrow[r] & H^1(X,\mathcal{O}) \arrow[dll, rounded corners] \\ & H^1(X,\Omega^1) \arrow[r] & H^2(X^{an},\mathbb{C}) \arrow[r] & 0 \end{tikzcd}$$

When X is projective, the snaky maps are 0. (This is hard!)

March 16 – Čech Cohomology. We introduced Čech cohomology.
Given a continuous map $\phi : U \to X$ and a sheaf $\mathcal{E}$ on U, the push forward $\phi_*\mathcal{E}$ on X is given by $(\phi_*\mathcal{E})(V) = \mathcal{E}(\phi^{-1}(V))$ for V open in X.

Given an open cover $X = \cup U_i$, write $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$, and write $\iota_{i_0\cdots i_p} : U_{i_0\cdots i_p} \hookrightarrow X$ for the inclusion. For a sheaf $\mathcal{E}$ on X, define $\mathcal{E}_{i_0\cdots i_p} = (\iota_{i_0\cdots i_p *})(\mathcal{E}|_{U_{i_0\cdots i_p}})$ so that $\mathcal{E}_{i_0\cdots i_p}(V) = \mathcal{E}(U_{i_0} \cap \cdots \cap U_{i_p} \cap V)$.

The Čech complex of $\mathcal{E}$ and U is $\mathcal{E} \to \prod \mathcal{E}_{i_0} \to \prod_{i_0<i_1} \mathcal{E}_{i_0 i_1} \to \cdots$, with the map from $\mathcal{E}_{i_0\cdots i_p} \to \mathcal{E}_{j_0\cdots j_{p+1}}$ given by

$$\begin{cases} 0 & \text{if } \{i_0,\cdots,i_p\} \not\subset \{j_0,\cdots,j_{p+1}\} \\ (-1)^r \rho^{U_{j_0\cdots j_{p+1}}}_{U_{j_0\cdots \widehat{j_r}\cdots j_{p+1}}} & \text{if } \{i_0,\cdots,i_p\} = \{j_0,\cdots,\widehat{j_r},\cdots,j_{p+1}\}. \end{cases}$$

Theorem: This is exact. (Hartshorne, Lemma III.4.2)

We have $H^q(X, \mathcal{E}_{i_0\cdots i_p}) = H^q(U_{i_0\cdots i_p}, \mathcal{E}|_{U_{i_0\cdots i_p}})$. If $\mathcal{E}|_{U_{i_0\cdots i_p}}$ is acyclic, then $H^q(X,\mathcal{E}) = H^q(\text{Čech complex})$.

Theorem (coming Friday): If U is affine and $\mathcal{E}$ is a quasi coherent sheaf on U, then $H^q(U,\mathcal{E}) = 0$ for $q > 0$.

So in particular if we have affine covers whose $(p+1)$-fold intersections are affine for all p and $\mathcal{E}$ is quasi coherent, then we can compute cohomology using Čech complexes. Note that for a separated scheme X, the intersection of affines is affine.

Example. Let X be a triangulated manifold, I the set of vertices of the triangulation. For $i \in I$, set U_i to be the union of the interiors of all faces (any dimension) containing i. Then $U_{i_0\cdots i_p} = \emptyset$ if $(i_0,\cdots,i_p)$ is not a face and $U_{i_0\cdots i_p}$ is contractible if $(i_0,\cdots,i_p)$ is a face. The Čech complex of $\mathbb{R}$ on X with respect to U_i is the cochain complex $C^\bullet(X)$ with respect to the triangulation. In particular, if $H^q(U,\mathcal{E})$ vanishes for U contractible, then $H^q(X,\mathcal{E})$ is $H^q(\text{Čech complex})$.

Example. $\mathbb{P}^1 = U \cup V$ for $U = \operatorname{Spec} k[x]$, $V = \operatorname{Spec} k[x^{-1}]$. Then $U \cap V = \operatorname{Spec} k[x,x^{-1}]$. The Čech complex for $\mathcal{O}$ with respect to the cover (U,V) is $\mathcal{O}(U) \oplus \mathcal{O}(V) \to \mathcal{O}(U \cap V) \to 0 \to 0 \to \cdots$, where the map $k[x] \oplus k[x^{-1}] \to k[x,x^{-1}]$ is given by $(f,g) \mapsto f - g$. So $H^0(\mathbb{P}^1,\mathcal{O}) = k$, $H^1(\mathbb{P}^1,\mathcal{O}) = 0$.

Example. Work with the sheaf Ω^1 on $\mathbb{P}^1$. $\Omega(U) \oplus \Omega(V) \to \Omega(U \cap V) \to \cdots$. The map $k[x]\cdot dx \oplus k[x^{-1}]\cdot x^{-2}dx \to k[x,x^{-1}]\cdot dx \to 0 \cdots$ is given by $(\alpha,\beta) \mapsto \alpha - \beta$. So $H^0(\mathbb{P}^1,\Omega) = 0$ and $H^1(\mathbb{P}^1,\Omega) = k \cdot x^{-1}dx$.

March 18 – Examples of Čech cohomology computations.

Example. Let us compute the Čech cohomology $H^\bullet(\mathbb{P}^2, \mathcal{O})$ of the structure sheaf $\mathcal{O}$ on $\mathbb{P}^2$. $\mathbb{P}^2$ is covered by 3 copies of $\mathbb{A}^2$. If we write $\mathbb{P}^2 = \operatorname{Proj} k[x, y, z]$, then the open sets are

$$\operatorname{Spec} k\left[\frac{x}{z}, \frac{y}{z}\right], \quad \operatorname{Spec} k\left[\frac{x}{y}, \frac{z}{y}\right], \quad \text{and } \operatorname{Spec} k\left[\frac{y}{x}, \frac{z}{z}\right].$$

Set $u = \frac{x}{z}$ and $v = \frac{y}{z}$, then the cover is

$$\operatorname{Spec} k[u, v], \quad \operatorname{Spec}\left[\frac{u}{v}, v^{-1}\right], \quad \text{and } \operatorname{Spec} k\left[\frac{v}{u}, u^{-1}\right].$$

The overlaps are

$$\begin{aligned}
\operatorname{Spec} k[u, v] \,\cap\, \operatorname{Spec} k\left[\frac{u}{v}, v^{-1}\right] &= \operatorname{Spec} k[u, v^{\pm}],\\
\operatorname{Spec} k[u, v] \,\cap\, \operatorname{Spec} k\left[\frac{v}{u}, u^{-1}\right] &= \operatorname{Spec} k[u^{\pm}, v],
\end{aligned}$$

and

$$\operatorname{Spec} k\left[\frac{u}{v}, v^{-1}\right] \,\cap\, \operatorname{Spec} k\left[\frac{v}{u}, u^{-1}\right] = \operatorname{Spec} k\left[\left(\frac{u}{v}\right)^{\pm}, u^{-1}\right].$$

Here, $u^{\pm}$ means that one must include both u and u^{-1}. The triple overlap is

$$\operatorname{Spec} k[u, v] \,\cap\, \operatorname{Spec} k\left[\frac{u}{v}, v^{-1}\right] \,\cap\, \operatorname{Spec} k\left[\frac{v}{u}, u^{-1}\right] = \operatorname{Spec} k[u^{\pm}, v^{\pm}].$$

Then, the Čech complex is

$$\begin{tikzcd}
k\left[u^{-1}, \frac{v}{u}\right] \arrow[r] \arrow[ddr] & k[u^{\pm}, v] \arrow[ddr] & \\
\oplus & \oplus & \\
k[u, v] \arrow[uur] \arrow[ddr] & k\left[\left(\frac{u}{v}\right)^{\pm}, u^{-1}\right] \arrow[r] & k[u^{\pm}, v^{\pm}] \\
\oplus & \oplus & \\
k\left[v^{-1}, \frac{u}{v}\right] \arrow[r] \arrow[uur] & k[v^{\pm}, u] \arrow[uur] &
\end{tikzcd}$$

We compute that $H^0(\mathbb{P}^2, \mathcal{O}) = k$ and $H^1(\mathbb{P}^2, \mathcal{O}) = H^2(\mathbb{P}^2, \mathcal{O}) = 0$.

Example. Let us compute Čech cohomology $H^\bullet(\mathbb{P}^2, \Omega^2)$. This time the Čech complex is

$$\begin{tikzcd}
k\left[u^{-1}, \frac{v}{u}\right] u^{-3} du \wedge dv \arrow[r] \arrow[ddr] & k[u^{\pm}, v] du \wedge dv \arrow[ddr] & \\
\oplus & \oplus & \\
k[u, v] du \wedge dv \arrow[uur] \arrow[ddr] & k\left[\left(\frac{u}{v}\right)^{\pm}, u^{-1}\right] u^{-3} du \wedge dv \arrow[r] & k[u^{\pm}, v^{\pm}] du \wedge dv \\
\oplus & \oplus & \\
k\left[v^{-1}, \frac{u}{v}\right] v^{-3} du \wedge dv \arrow[r] \arrow[uur] & k[v^{\pm}, u] du \wedge dv \arrow[uur] &
\end{tikzcd}$$

We compute that $H^0(\mathbb{P}^2, \Omega^2) = H^1(\mathbb{P}^2, \Omega^2) = 0$ and $H^2(\mathbb{P}^2, \Omega^2) = k \cdot \frac{du \wedge dv}{uv} \simeq k$.

Example. The hyperelliptic curve X is constructed by glueing the affine schemes

$$A = \operatorname{Spec}\left(\frac{k[x,y]}{(y^2 = f_{2g+1}x^{2g+1} + \ldots + f_1 x)}\right) \text{ and } B = \operatorname{Spec}\left(\frac{k[u,v]}{(v^2 = f_{2g+1}u + \ldots + f_1 u^{2g+1})}\right),$$

where we glue $\{x \neq 0\}$ to $\{u \neq 0\}$ by $u = x^{-1}$ and $v = x^{-g-1}y$. Observe that

$$\frac{k[x,y]}{(y^2 = f_{2g+1}x^{2g+1} + \ldots + f_1 x)} = k[x] \oplus k[x] \cdot y \text{ and } \frac{k[u,v]}{(v^2 = f_{2g+1}u + \ldots + f_1 u^{2g+1})} = k[u] \oplus k[u] \cdot v.$$

The Čech complex is $\mathcal{O}(A) \oplus \mathcal{O}(B) \to \mathcal{O}(A \cap B)$, which becomes

$$(k[x] \oplus k[x] \cdot y) \oplus \left(k[x^{-1}] \oplus k[x^{-1}] \cdot x^{-g-1}y\right) \to k[x^{\pm}] \oplus k[x^{\pm}] \cdot y.$$

This breaks up into 2 complexes:

$$k[x] \oplus k[x^{-1}] \to k[x^{\pm}] \text{ and } k[x] \cdot y \oplus k[x^{-1}]x^{-g-1}y \to k[x^{\pm}]y.$$

We compute that $H^0(X, \mathcal{O}) = k$ and $H^1(X, \mathcal{O}) \simeq k^g$, with basis $yx^{-1}, yx^{-2}, \ldots, yx^{-g}$.

March 20 – Cohomology of quasi-coherent sheaves vanishes on affines. The goal of today was to to show that if $X = \operatorname{Spec} R$, for R a noetherian ring, and $\mathcal{E}$ is a quasi-coherent sheaf on X, then $H^q(X, \mathcal{E}) = 0$ for $q > 0$.

The key to this result is the correspondence between quasi-coherent sheaves on $\operatorname{Spec} R$ and $R-$modules. We recall that given a R-module M we can construct the sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ with the property that $\widetilde{M}(D(f)) = f^{-1}M$. We stated a while ago that every quasi-coherent sheaf on $\operatorname{Spec} R$ is of the form $\widetilde{M}$; today we will finally prove this.

We proved the following key lemmas.

Lemma (Fact 1)**.** If $u \in \mathcal{E}(x)$ and for $f \in R$ we have that $u|_{D(f)} = 0$, then there exist some N such that $f^N u = 0$.

Lemma (Fact 2)**.** If $f \in R$ and $U \in \mathcal{E}(D(f))$, then there exist N and $v \in \mathcal{E}(X)$ such that $f^N u = v|_{D(f)}$.

These two lemmas allow us to prove the following theorem.

Theorem. Let $\mathcal{E}$ be a quasi-coherent sheaf on $X = \operatorname{Spec} R$. Then $\mathcal{E} \cong \widetilde{\mathcal{E}(X)}$.

We prove the theorem by considering the map $f^{-1}\mathcal{E}(X) \to \mathcal{E}(D(f))$ whose existence is granted by the universal property of localization. Moreover, since this map is defined on a basis of the topology, it extends to a map $\widetilde{\mathcal{E}(X)} \to \mathcal{E}$. Finally, Lemma 1 gives us injectivity of the map and Lemma 2 gives us surjectivity.

Next, we applied this result and functoriality of $\sim$ to prove the following theorem.

Theorem. Let $X = \operatorname{Spec} R$ and consider

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

a SES of quasi-coherent sheaves on X. Then

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0,$$

is exact.

We proved the theorem by considering $D = \mathrm{CoKer}(\mathcal{B}(X) \to \mathcal{C}(X))$. Since $\mathcal{C}(X) \to D$ is a surjection, $\mathcal{C} \to \widetilde{D}$ is a surjection. So the composite $\mathcal{B} \to \mathcal{C} \to \mathcal{D}$ is a surjection. But this composite is 0, so $\mathcal{D} = 0$ and $D = 0$. $\square$

Finally, we used these results to establish our initial goal.

Theorem. Let $X = \operatorname{Spec} R$ and consider $\mathcal{E}$ a quasi-coherent sheaf on X. Then $H^q(X, \mathcal{E}) = 0$ for $q \geq 1$.

Proof by induction on q. Inject $\mathcal{E}$ into a quasi-coherent injective $\mathcal{I}$, giving the short exact sequence $0 \to \mathcal{E} \to \mathcal{I} \to \mathcal{F} \to 0$. Then we have an exact sequence $0 \to H^0(\mathcal{E}) \to H^0(\mathcal{I}) \to H^0(\mathcal{F}) \to H^1(\mathcal{E}) \to 0$. But we showed in the previous theorem that $H^0(\mathcal{E}) \to H^0(\mathcal{I})$ is surjective, so $H^1(\mathcal{E}) = 0$. For the inductive case, note that the long exact sequence gives $H^{q+1}(\mathcal{E}) \cong H^q(\mathcal{F})$. $\square$

March 23 – Eventual global generation. Today we explained how to use line bundles to extend yesterday's ideas to non-affine schemes. Let X be a scheme (not necessarily affine) and $\mathcal{L}$ a line bundle on X. Let $\phi \in \mathcal{L}(X)$ (analogous to yesterday's f), and let U be the open set $U = \{x \in X : \phi(x) \neq 0\} \subseteq X$. Finally, let $\mathcal{E}$ be a quasi-coherent sheaf on X. We have the following results, analogous to Friday's Fact 1 and Fact 2.

Lemma. If $v \in \mathcal{E}(X)$ obeys $v|_U = 0$, then there exists $N > 0$ such that $\phi^N v = 0$ in $\mathcal{L}^{\otimes N} \otimes \mathcal{E}$.

Lemma. Let X, $\mathcal{E}$ and $\mathcal{L}$ be as above and let $u \in \mathcal{E}(U)$. Then there exists N and $v \in \mathcal{L}^{\otimes N} \otimes \mathcal{E}(X)$ such that $\phi^N u = v|_U$ in $\mathcal{L}^{\otimes N} \otimes \mathcal{E}(U)$.

Theorem. Let X be Noetherian and Proper over an algebraically closed field. Let $\mathcal{E}$ be quasi-coherent and let $\mathcal{L}$ be an ample sheaf on X. Then for some $N > 0$, the tensor product $\mathcal{E} \otimes \mathcal{L}^{\otimes N}$ is globally generated.

Proof sketch. Replace $\mathcal{L}$ by $\mathcal{L}^{\otimes d}$ we may assume $\mathcal{L}$ is very ample and thus $i : X \hookrightarrow \mathbb{P}^r$ and $\mathcal{L} = i^*\mathcal{O}(1)$. Since X is proper, $i(X)$ is closed in $\mathbb{P}^r$. Fix $x \in X$; let $\phi \in H^0(\mathbb{P}^r, \mathcal{O}(1))$ define a hyperplane not through x.

Let U be the open set in X where $\phi \notin 0$. Now $\mathbb{P}^r \backslash \{\phi = 0\} = \mathbb{A}^r$ and U is closed in $\mathbb{A}^r$ so U is affine. So $\mathcal{E}_x$ is generated as an $\mathcal{O}_x$−module by $\mathcal{E}(U)$ as $\mathcal{E}$ is coherent, pick generators $v_1, \ldots, v_m$ for $\mathcal{E}(U)$. We can lift $\phi^N v_1, \ldots, \phi^N v_m$ to $H^0(X, \mathcal{E} \otimes \mathcal{L}^{\otimes N})$ by prop 2.

This proves that some N works at x, then one uses Nakayama to deduce that N works in a neighborhood of x. Finish by compactness. $\square$

Corollary. Let X be proper and Noetherian, let $\mathcal{L}$ be an ample sheaf on X and $\mathcal{E}$ a coherent sheaf. Then there exists a surjection $(\mathcal{L}^{\otimes(-N)})^{\oplus M} \twoheadrightarrow \mathcal{E}$.

March 25 – Cohomology of line bundles on projective spaces. Today's goal is to compute the cohomology groups of line bundles on projective spaces. First we go through the restatement of the concepts in last class: Let X be proper scheme, $\mathcal{L}$ be an ample line bundle, and $\mathcal{E}$ is a coherent sheaf on X, then $\mathcal{E} \otimes \mathcal{L}^{\otimes N}$ is globally generated in the sense that $\Gamma(\mathcal{E} \otimes \mathcal{L}^{\otimes N})$ generate $(\mathcal{E} \otimes \mathcal{L}^{\otimes N})_x$ for all $x \in X$. In other words, we have the surjection $\Gamma(\mathcal{E} \otimes \mathcal{L}^{\otimes N}) \otimes \mathcal{O} \to \mathcal{E} \otimes \mathcal{L}^{\otimes N}$, in which $\Gamma(\mathcal{E} \otimes \mathcal{L}^{\otimes N}) \oplus \mathcal{O} \cong \mathcal{O}^{\oplus M}$ for some M. So if we have lots of sections $\sigma_1, \cdots, \sigma_M$ of $\mathcal{E} \otimes \mathcal{L}^{\otimes N}$, then we can use these sections to get a map from $\mathcal{O}^{\oplus M}$ to $\mathcal{E} \otimes \mathcal{L}^{\otimes N}$ by sending M−tupe regular functions $(f_1, \cdots, f_M)$ to $\sum f_i \sigma_i$. Alternatively, we can say $(\mathcal{L}^{-\otimes N})^M \to \mathcal{E}$ is surjective.

Example (deg 3 plane curve)**.** Let C be the curve $x^3+y^3+z^3=0$ in $\mathbb{P}^2$, $\mathcal{O}_C$ be the structure sheaf of C, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-C) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

where the line bundle $\mathcal{O}_{\mathbb{P}^2}(-C)$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C$. It turns out that $\mathcal{O}_{P^2}(-C) \cong \mathcal{O}(-3)$. More explicitly, on $\operatorname{Spec} k[\frac{x}{z},\frac{y}{z}]$, we see $\mathcal{O}(-C)$ is a $k[\frac{x}{z},\frac{y}{z}]$-module generated by $1+\frac{x^3}{z^3}+\frac{y^3}{z^3}$, and $\mathcal{O}(-3)$ is a $k[\frac{x}{z},\frac{y}{z}]$-module generated by z^{-3}, and this isomorphism sends generator to generator. The above exact sequence can thus be rewritten as

$$0 \longrightarrow \mathcal{O}(-3) \xrightarrow{\cdot(x^3+y^3+z^3)} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Now we are going to compute $H^q(\mathbb{P}^n,\mathcal{O}(d))$. Note that all line bundles on $\mathbb{P}^n$ are equivalent to $\mathcal{O}(d)$ for some d. As $\mathbb{P}^n$ is smooth (hence locally factorial), the Cartier divisors are isomorphic to the Weil divisors modulo rational equivalence. We have the following results (with respect to the open cover $U_j=\{x_j\neq 0\}$):

$$\begin{aligned}
H^0(\mathbb{P}^n,\mathcal{O}(d)) &= k\cdot\{x_0^{d_0}\cdots x_n^{d_n} : d_0,\cdots,d_n \geqslant 0, \Sigma d_i = d\},\\
H^q(\mathbb{P}^n,\mathcal{O}(d)) &= 0 \text{ for } 0<q<n\\
H^n(\mathbb{P}^n,\mathcal{O}(d)) &= k\cdot\{x_0^{d_0}\cdots x_n^{d_n} : d_0,\cdots,d_n < 0, \Sigma d_i = d\}.
\end{aligned}$$

Note that $H^0(\mathbb{P}^n,\mathcal{O}(d))$ is nontrivial only for $d\geq 0$, and $H^n(\mathbb{P}^n,\mathcal{O}(d))$ is nontrivial only for $d\leq -n-1$.

We use the Čech cover $\{x_j\neq 0\}$. Fix a degree $(d_0,\cdots,d_n)\in\mathbb{Z}^{n+1}$ with $\Sigma d_i=d$, and set $N=\{j : d_j<0\}\subseteq\{0,\cdots,n\}$. Then we have the following relation:

$$x_0^{d_0}\cdots x_n^{d_n}\in\mathcal{O}(d)(U_{j_0\cdots j_p}) \Longleftrightarrow N\subseteq\{j_0,\cdots,j_p\}$$

If $N=\emptyset$, then we get the Čech complex

$$k^{n+1}\to k^{\binom{n+1}{2}}\to\cdots\to k^{\binom{n+1}{n+1}},$$

which only contributes to H^0. Another boundary case is $N=\{0,1,\cdots,n\}$, then the complex becomes

$$0\to 0\to\cdots\to k,$$

which only contributes to H^n. In all other cases, we get

$$0\to\cdots 0\to \underbrace{k}_{\#N-th}\to k^{n-\#N}\to k^{n-\#N}\to\cdots\to k,$$

which gives $H^q=0$ for every degree except the missed degenerate cases.

Lastly, we talk about an application to the cubic curve. From the short exact sequence discussed above, we get the long exact sequence as follows:

$$\begin{tikzcd}
0 \arrow[r] & H^0(P^2,\mathcal{O}(-3)) \arrow[r] & H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}) \arrow[r] & H^0(C,\mathcal{O}_C) \arrow[dll, out=-10, in=170] \\
& H^1(P^2,\mathcal{O}(-3)) \arrow[r] & H^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}) \arrow[r] & H^1(C,\mathcal{O}_C) \arrow[dll, out=-10, in=170] \\
& H^1(P^2,\mathcal{O}(-3)) \arrow[r] & H^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}) \arrow[r] & 0.
\end{tikzcd}$$

Using the results on cohomology of line bundles, we have

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & k \arrow[r] & H^0(C,\mathcal{O}_C) \arrow[dll] \\
 & 0 \arrow[r] & 0 \arrow[r] & H^1(C,\mathcal{O}_C) \arrow[dll] \\
 & k \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd}$$

from which we deduce that $H^0(C,\mathcal{O}_C) \cong k$ and $H^1(C,\mathcal{O}_C) \cong k$.

March 27 – Serre Vanishing. Our goal today will be to prove the Serre vanishing theorem and some related things. Let X be projective over k,[4] $\mathcal{E}$ be coherent on X and $\mathcal{L}$ be ample. We're going to show that $\dim_k H^q(X,\mathcal{E}) < \infty$ for all q, and that for N sufficiently large, $H^q(X,\mathcal{E}\otimes\mathcal{L}^{\otimes N}) = 0$ for $q \geq 1$ (this is called Serre's vanishing theorem).

First, we should do some clean-up: we can replace $\mathcal{L}$ by $\mathcal{L}^{\otimes d}$ and assume we have an embedding $X \hookrightarrow \iota\mathbb{P}^r$ with $\mathcal{L} = \mathcal{O}(1)\big|_X$. Replacing $\mathcal{E}$ by $i_*\mathcal{E}$, we can assume $X = \mathbb{P}^r$. The first thing we'll need is:

Theorem. (Grothendieck) If $q > \dim X$, then $H^q(X,\mathcal{E}) = 0$.

Remark. Grothendieck (and Hartshorne) proved this with very minimal hypotheses: X needs to be a Noetherian space, and $\mathcal{E}$ can be any sheaf of abelian groups. For $X = \mathbb{P}^r$, this is much easier: we cover with $r+1$ charts. Incidentally, this is not so bad even when X is merely projective (and not $\mathbb{P}^r$ itself).

If $X \subset \mathbb{P}^r$ is closed (and $|k| = \infty$), choose $\lambda_1 \in H^0(\mathbb{P}^r,\mathcal{O}(1))$ so that the set given by $\lambda_1 = 0$ contains no component of X. Set $X_2 = X \cap V(\lambda_1)$, so $\dim X_2 < \dim X$. We repeat, choosing λ_2 so that $V(\lambda_2)$ contains no component of X_2, and construct $X_3 = X_2 \cap V(\lambda_2)$. We get $V(\lambda_1,\dots,\lambda_{\dim X+1}) \cap X = \varnothing$, and then $U_i = X \cap D(\lambda_i)$ is an affine cover.

We now proceed by reverse induction on q. If $q > r$, we are done. Now, from Monday, we had a surjection:

$$\mathcal{O}(-m_1)^{\oplus m_2} \to \mathcal{E}$$

for M_1, M_2 sufficiently large. So we get a short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{O}(-m_1)^{\oplus m_2} \to \mathcal{E} \to 0$$

Then from the long exact sequence, we get that

$$H^q(\mathcal{O}(-m_1)^{\oplus m_2} \to H^q(\mathcal{E}) \to H^q(\mathcal{F})$$

is exact. The leftmost term is finite dimensional, by the computations from Wednesday. The rightmost term is finite dimensional by reverse induction, so $\dim_k H^q(\mathcal{E}) < \infty$.

For Serre's vanishing theorem, note that

$$0 \to \mathcal{F}\otimes\mathcal{O}(N) \to \mathcal{O}(N-m_1)^{\oplus m_2} \to \mathcal{E}\otimes\mathcal{O}(N) \to 0$$

[4]though we can do this with a more interesting base

is also exact. We get:

$$\begin{array}{ccccc} H^q(\mathcal{O}(N-m_1)^{\oplus m_2}) & \longrightarrow & H^q(\mathcal{E}(N)) & \longrightarrow & H^{q+1}(\mathcal{F}(N)) \\ \Big\| \text{for } N \gg 0, \text{ by Wed.} & & & & \Big\| \text{for } N \gg 0, \text{ by induction} \\ 0 & & & & 0 \end{array}$$

for $q \geq 1$. So $H^q(\mathcal{E}(N)) = 0$ for N sufficiently large, which completes our proof.

March 30 – Hilbert Series. We work on projective space over a field. Let $\mathcal{A}$ be a coherent sheaf on $\mathbb{P}^r$ (or anything projective over k). Define $\chi(\mathcal{A}) = \sum_q (-1)^q \mathrm{dim} H^q(\mathbb{P}^r, \mathcal{A})$. From Friday's lecture, this is a sum of finitely many finite terms.

Proposition. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence, then $\chi(\mathcal{B}) = \chi(\mathcal{A}) + \chi(\mathcal{C})$.

Proof. Take the alternating sum of dimensions in the long exact sequence. □

Set $\mathrm{Hilb}_{\mathcal{A}}(n) := \chi(\mathcal{A}(n))$, so the short sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is exact. Then $0 \to \mathcal{A}(n) \to \mathcal{B}(n) \to \mathcal{C}(n) \to 0$ is exact, so we have $\mathrm{Hilb}_{\mathcal{B}}(n) = \mathrm{Hilb}_{\mathcal{A}}(n) + \mathrm{Hilb}_{\mathcal{C}}(n)$. By Serre vanishing, for $n \gg 0$, $\mathrm{Hilb}_{\mathcal{C}}(n) = \mathrm{dim} H^0(\mathcal{C}(n))$.

Proposition. $\mathrm{Hilb}_{\mathcal{C}}(n)$ is a polynomial in n, $\deg \leq r$.

Proof. Induct on r. The base case is $r = 0$. The projective space $\mathbb{P}^0$ is simply a point, $\mathcal{C}$ is a vector space and $\mathrm{Hilb}_{\mathcal{C}}(n) =$ dim of that vector space $=$ const.

Embed $\mathbb{P}^{r-1}$ in $\mathbb{P}^r$ as the hyperplane $\{z_r = 0\}$. Then we have a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \xrightarrow{\cdot z_r} \mathcal{E}(1) \to \mathcal{C} \to 0$$

where $\mathcal{K}$ and $\mathcal{C}$ are the kernel and cokernel of multiplication by z_r. Note that $\mathcal{K}$ and $\mathcal{C}$ are supported on $\mathbb{P}^{r-1}$, so they are pushed forward from schemes supported on $\mathbb{P}^{r-1}$, and we have

$$\mathrm{Hilb}_{\mathcal{E}(1)}(n) - \mathrm{Hilb}_{\mathcal{E}}(n) = (\text{polynomial of degree} \leq r-1).$$

The left hand side is $\mathrm{Hilb}_{\mathcal{E}}(n+1) - \mathrm{Hilb}_{\mathcal{E}}(n)$, so $\mathrm{Hilb}_{\mathcal{E}}(n)$ is a polynomial of degree $\leq r$ as desired. □

The degree of the Hilbert polynomial is easy to explain: It is $\dim \mathrm{Support}(\mathcal{E})$. Call this d. If $\mathcal{E} = \mathcal{O}_Z$, the leading term of the Hilbert polynomial is $(\deg Z) n^d/d!$. Similarly, if $\mathcal{E}$ is a rank s vector bundle on $\mathbb{P}^r$, then the leading term is $sn^r/r!$. These facts can be proved by Noether normalization or repeated slicing.

April 1 – Hilbert series for many line bundles. Let X be projective over k, and let $L_1, \ldots, L_s$ be line bundles on X. For a coherent sheaf $\mathcal{E}$ on X, write

$$\mathcal{E}(a_1, \ldots, a_s) = \mathcal{E} \otimes L_1^{a_1} \otimes \ldots \otimes L_s^{a_s}$$
$$h_{\mathcal{E}}(a_1, \ldots, a_s) = \chi(\mathcal{E}(a_1, \ldots, a_s))$$

Today, we'll show that $h_{\mathcal{E}}$ is a polynomial on $\mathbb{Z}^s$.

Before we start this, we will explore a consequence of Serre's vanishing theorem.

Lemma. Let X be projective over k, let L be any line bundle on X, and let H be an ample line bundle on X. Then $L \otimes H^{\otimes N}$ is ample for N sufficiently large.

Proof. See Proposition III.5.3 in Hartshorne. The moral of the proof is that ample sheaves give us room to do what we need: we see that they 'separate points' (i.e. we get a surjection $H^0(L) \twoheadrightarrow k \oplus k$) and that they 'separate tangent vectors' (i.e. we get a surjection $H^0(L) \twoheadrightarrow L_x/\mathfrak{m}_x^2 L_x$). □

This lemma provides evidence for the idea that the ample line bundles form an 'open cone' in $\mathrm{Pic}(X)$, as "sliding a line bundle L in the direction of an ample line bundle H eventually lands one among the ample line bundles". Some other evidence for this idea includes are that ample line bundles are closed under tensor products (i.e. "closed under addition"), and that if $H^{\otimes N}$ is ample, then H is also ample (i.e. "closed under positive rescaling"). If we allow ourselves to formally rescale by any positive rational number, we can rephrase the lemma as saying: If H is ample, and D is anything, then $H + (1/N)D$ is ample for large N, making it look even more like an open condition. This idea of defining ampleness for elements of $\mathrm{Pic}\, X \otimes \mathbb{R}$ can be made precise, and is common in birational geometry.

Now, back to our original setting.

Proposition. Let X be projective over k, $L_1, \dots, L_s$ be line bundles, and $\mathcal{E}$ be a coherent sheaf on X. Then

$$(a_1, \dots, a_s) \mapsto \chi(\mathcal{E} \otimes L_1^{\otimes a_1} \otimes \ldots \otimes L_s^{\otimes a_s})$$

is a polynomial on $\mathbb{Z}^s$.

Proof. Let H be ample, then by replacing $(L_1, \dots, L_s)$ by $(L_1 + nH, \dots, L_s + nH)$, we may assume that the L_i's are very ample. We proceed by induction on both s and $\dim \mathrm{Support}(\mathcal{E})$. Choose a section z of L_s not containing the generic point of any component of $\mathrm{Support}(\mathcal{E})$, then there is a short exact sequence

$$0 \to \mathcal{E} \otimes L_s^{-1} \xrightarrow{\cdot z} \mathcal{E} \to \mathcal{E}/z\mathcal{E} \to 0$$

It follows that

$$h_{\mathcal{E}}(a_1, \dots, a_s) - h_{\mathcal{E}}(a_1, \dots, a_{s-1}, a_s - 1) = h_{\mathcal{E}/z\mathcal{E}}(a_1, \dots, a_s).$$

Then, $h_{\mathcal{E}/z\mathcal{E}}(a_1, \dots, a_s)$ is polynomial by induction on $\dim \mathrm{Support}(\mathcal{E})$, and $h(a_1, \dots, a_{s-1}, 0)$ is polynomial by induction on s. Consequently, we can write

$$\begin{aligned} h(a_1, \dots, a_{s-1}, a_s) &= h(a_1, \dots, a_{s-1}, 0) + \sum_{k=1}^{a_s} h(a_1, \dots, a_{s-1}, k) - h(a_1, \dots, a_{s-1}, k-1) \\ &= \sum_{k=1}^{a_s} \mathrm{poly}(a_1, \dots, a_{s-1}, k) + \mathrm{poly}(a_1, \dots, a_{s-1}) \end{aligned}$$

□

April 3 – Hilbert polynomials and intersection theory. Today we talked about the leading term of the Hilbert polynomial in several variables. Throughout the day we considered X, a projective scheme over a field k, and $L_1, \dots, L_s$ line bundles over X. For $\mathcal{E}$ a coherent sheaf, recall the notation $\mathcal{E}(a_1, \dots, a_s) = \mathcal{E} \otimes L_1^{a_1} \otimes \cdots \otimes L_s^{a_s}$.

Last time we introduced the polynomial $h_{\mathcal{E}}(a_1, \dots, a_s) = \chi_{\mathcal{E}}(a_1, \dots, a_s)$. We usually study the leading term of this polynomial when $\mathcal{E} = \mathcal{O}_X$ and X is a surface. From Riemann-Roch, we already know that when X is a curve, we have that $h_{\mathcal{O}_X}(a_1, \dots, a_s) = \sum a_j \deg L_j - g + 1$, so that the leading term is given by the degree.

Let us consider a surface X with line bundles L_j , where we let $L_j = \mathcal{O}(D_j)$, for some divisors D_j. If we consider two line bundles, then

$$h_{\mathcal{O}_X}(a_1, a_2) = \chi(L_1^{a_1} \otimes L_2^{a_2}) = \chi(a_1 D_1 + a_2 D_2) = \sum_{i,j=0,1,2} c_{ij} a_1^i a_2^j.$$

We know that $c_{00} = \chi(\mathcal{O}_X)$. Let us study c_{11}. We notice that

$$c_{11} = h(a_1, a_2) - h(a_1 - 1, a_2) - h(a_1, a_2 - 1) + h(a_1 - 1, a_2 - 1).$$

Using the SES

$$0 \to L_1^{-1} \to \mathcal{O} \to \mathcal{O}_{D_1} \to 0,$$

we can deduce that $h(a_1, a_2) - h(a_1 - 1, a_2) = h_{\mathcal{O}_{D_1}}(a_1, a_2)$ and also that $h(a_1, a_2 - 1) - h(a_1 - 1, a_2 - 1) = h_{\mathcal{O}_{D_1}}(a_1, a_2 - 1)$. Hence we need to study the difference $h_{\mathcal{O}_{D_1}}(a_1, a_2) - h_{\mathcal{O}_{D_1}}(a_1, a_2 - 1)$. Let us consider z a section of L_2 not vanishing on any component of D_1. If L_2 is very ample, such a section exists. Moreover, let us consider D_2 to be the divisor defined by the vanishing of z and let $D_1 \cap D_2$ be the scheme-theoretic intersection of these two divisors. Then multiplication by z gives us the short exact sequence

$$0 \to \mathcal{O}_{D_1} \otimes L_1^{a_1} \otimes L_2^{a_2 - 1} \to^{\cdot z} \mathcal{O}_{D_1} \otimes L_1^{a_1} \otimes L_2^{a_2} \to \mathcal{O}_{D_1 \cap D_2} \otimes L_1^{a_1} \otimes L_2^{a_2} \to 0.$$

Now $D_1 \cap D_2$ is 0-dimensional so that

$$c_{11} = \chi(\mathcal{O}_{D_1 \cap D_2} \otimes L_1^{a_1} \otimes L_2^{a_2}) = \chi(\mathcal{O}_{D_1 \cap D_2}) = \#(D_1 \cap D_2),$$

where the points of intersections are counted with multiplicity.

A similar discussion holds for the other coefficients of $h_{\mathcal{O}_X}$ for any number of line bundles s, so that

$$h_{\mathcal{O}_X}(a_1, \ldots, a_s) = \sum_{i,j} \# \frac{(D_i \cap D_j)}{2} a_i a_j + (\text{lower order terms}).$$

We remarked that when $a_i = a_j$ this formula gives us a symmetric bilinear form $PicX \times PicX \to \mathbb{Z}$ defined by $< \mathcal{O}(D_1), \mathcal{O}(D_2) >= \#(D_1 \cap D_2)$, where the intersection points are counted with multiplicity and D_1, D_2 have no common component. Moreover, it is interesting to notice that this formula generalises in the obvious way for schemes of dimension d.

We concluded the class with a few examples. Letting $X = \mathbb{P}^r$, we have that

$$\chi(\mathcal{O}(n)) = \binom{n+r}{r} = \frac{n^r}{r!} + (\text{lower order terms}).$$

In particular, this shows that r hyperplanes in $\mathbb{P}^r$ intersect in 1 point. For $X = \mathbb{P}^1 \times \mathbb{P}^1$, the symmetric quadratic form can be represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This example allows us to construct divisors can negative self intersection. More interestingly, we can use the Hilbert polynomial to show that the exceptional divisor in the blow up of $\mathbb{P}^2$ at a point has self intersection -1.

April 6 – Finite flat families. Today we discuss finite flat morphisms, and intuition for them. We first remind ourselves about finite morphisms, which we learned about last semester.

Recall that $\pi\colon Y \to X$ is ***finite*** if for every open affine $\operatorname{Spec} A \subset X$, its inverse image $\pi^{-1}(\operatorname{Spec} A)$ equals $\operatorname{Spec} B$, and the induced ring homomorphism $\pi^*\colon A \to B$ is module-finite. Recall that by Ex. II.3.4, it suffices to check this property on an affine cover. Recall also that finite morphisms are closed by Ex. II.3.5(b), and that finiteness is closed under base change, i.e., if π is finite in the cartesian square below, so is π':

$$\begin{tikzcd} X' \times_X Y \arrow[r] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\pi"] \\ X' \arrow[r] & X \end{tikzcd}$$

In the particular case when $X' = \{x\}$ a point in X, we defined the ***scheme-theoretic fibre*** at x, denoted $\pi^{-1}(x)$, to be the fibre product

$$\begin{tikzcd} \pi^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\pi"] \\ \operatorname{Spec} k(x) \arrow[r] & X \end{tikzcd}$$

and defined the ***length*** of the fibre as $\ell(\pi^{-1}(x)) = \dim_{k(x)} \mathcal{O}(\pi^{-1}(x)) = \dim_{k(x)} \pi_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} k(x)$. Last semester, we proved the following for varieties over algebraically closed fields:

Theorem. Let $\pi\colon Y \to X$ be a finite morphism of locally noetherian schemes. Then, $x \mapsto \ell(\pi^{-1}(x))$ is an upper semicontinuous function on X.

In commutative-algebraic language, if $\operatorname{Spec} A$ is an open affine subset of X, with preimage $\operatorname{Spec} B$, and if $\mathfrak{p} \subset A$ is a prime ideal, then $\ell(\pi^{-1}(\mathfrak{p})) = \dim_{\operatorname{Frac} A/\mathfrak{p}A} B \otimes_A \operatorname{Frac}(A/\mathfrak{p}A)$.

For examples, refer back to the Updates on 10/06 and 10/08 from last semester, and for the proof of the Theorem, refer to the Update on 10/10. The Theorem is also Ex. II.5.8(a).

Now, let's introduce flatness:

Theorem-Definition. Let $\pi\colon Y \to X$ be a finite morphism of locally noetherian schemes, where X is reduced. Then, the following are equivalent:

(1) $\ell(\pi^{-1}(x))$ is locally constant;
(2) the $\mathcal{O}_X$-module $\pi_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module;
(3) the $\mathcal{O}_X$-module $\pi_*\mathcal{O}_Y$ is flat as an $\mathcal{O}_X$-module.

In this case, we say that π is ***finite and flat***.

Proof of (1) $\Leftrightarrow$ (2). $\Leftarrow$. Let $x \in X$ such that $(\pi_*\mathcal{O}_Y)_x \cong \mathcal{O}_{X,x}$. Then by Ex. II.5.7($a$), there is an open subset $U \ni x$ on which $(\pi_*\mathcal{O}_Y)|_U \cong \mathcal{O}_U$.

$\Rightarrow$. Let $v_1, \ldots, v_r$ be r generators of $\pi_*\mathcal{O}_Y \otimes_{\mathcal{O}_X} k(x)$. Then by Nakayama's lemma, these generators lift to generators of $\mathcal{O}_Y$, and we have a surjection $\varphi\colon \mathcal{O}_U^{\oplus r} \to \pi_*\mathcal{O}_Y$ after possibly passing to an open neighborhood of x. We want to show this map is injective. Passing to affine opens $\operatorname{Spec} A \subset X$ and $\operatorname{Spec} B = \pi^{-1}(\operatorname{Spec} A) \subset Y$, we have that $A^r \otimes_A k(\mathfrak{p}) \to B \otimes_A k(\mathfrak{p})$ is a vector space isomorphism for every $\mathfrak{p} \in \operatorname{Spec} A$. But this implies $\ker \varphi \subset \bigcap_{\mathfrak{p} \subset A} \mathfrak{p}A^r$. Since X is reduced, $\ker \varphi = 0$. $\square$

Note that (1) $\Leftarrow$ (2) did not use the reduced hypothesis, while it is necessary in (1) $\Rightarrow$ (2). For example, if $Y = \operatorname{Spec} k$ and $X = \operatorname{Spec} k[x]/x^2$, with $\operatorname{Spec} k \to \operatorname{Spec} k[x]/x^2$ the obvious map and $p \in X$ the unique point, then $\pi_*\mathcal{O}_Y = \widetilde{k}$ is not locally free on X, even though $\ell(\pi^{-1}(p)) = 1$.

Proof of (2) $\Leftrightarrow$ (3). (not done in class). (2) $\implies$ (3) Flat is a local condition, and free clearly implies flat, so locally free implies flat.

(3) $\implies$ (2): Again, the question is local, so we may assume that $X = \operatorname{Spec} A$ for a local ring A with maximal ideal $\mathfrak{m}$ and $A/\mathfrak{m} = k$. Let $Y = \operatorname{Spec} B$. Pick generators v_1, v_2, …, v_r for the (necessarily finite dimension) k vector space $B/\mathfrak{m}B$, and lift them to u_1, u_2, …, $u_r \in B$. Then we have a map $A^r \to B$ sending the j-th generator of A^r to u_r. This map is surjective by Nakayama. Let the kernel be K, so we have $0 \to K \to A^r \to B \to 0$. Tensoring with k, we have the exact sequence $\operatorname{Tor}_1^A(B,k) \to K/\mathfrak{m}K \to k^r \to B/\mathfrak{m}B \to 0$. But $\operatorname{Tor}_1^A(B,k) = 0$ since B is flat, and $k^r \to B/\mathfrak{m}B$ is an isomorphism by construction, so $K/\mathfrak{m}K = 0$. By Nakayama, this shows that $K = 0$. So $A^r \cong B$. □

April 8 – Flatness. Given a commutative ring A, and A−modules M, X, Y, Z, with s.e.s $0 \to X \to Y \to Z \to 0$, then we get an exact sequence

$$M \otimes X \to M \otimes Y \to M \otimes Z \to 0$$

However, we may not have injectivity in the left map.

Example. Let A be a domain, $f \in A$ and M an A-module with f−torsion. Then $0 \to A \xrightarrow{f} A \to A/fA \to 0$ is exact, and $M \xrightarrow{f} M \to M/fM \to 0$ is exact, but we don't have injectivity on the LHS.

Definition. M is called ***flat*** if whenever X injects into Y, the map $M \otimes X \to M \otimes Y$ is an injection.

Proposition. M flat over a domain $\Rightarrow$ M is torsion free. Flatness over a PID, or more generally, over a Dedekind domain $\iff$ Torsion free.

Proof. It is enough to show that for $I \subseteq A$, $I \otimes M \to M$ is injective. For a PID, any $I = (f)$, thus, $I \otimes M = (f) \otimes M \to M$, map is given by $af \otimes m \mapsto af \cdot m$. And this map is injective if and only if M has no f−torsion. □

Proposition. Flatness is local, i.e., if $\forall \mathcal{P} \in \operatorname{Spec} A$, $M_{\mathcal{P}}$ is $A_{\mathcal{P}}$−flat, then M is A-flat.

Proof sketch. Injectivity is local. □

Also, $S^{-1}A$ is always A flat and $S^{-1}M$ is $S^{-1}A$-flat if M is A-flat.

For M, N A-modules, there exists a sequence of modules $\operatorname{Tor}_i^A(M,N)$ where $i \in \mathbb{N}$. $\operatorname{Tor}_0(M,N) = M \otimes N$. $\forall$ s.e.s $0 \to X \to Y \to Z \to 0$, we have the following l.e.s:

$$\begin{tikzcd}
Tor_2(M,X) \arrow[r] & Tor_2(M,Y) \arrow[r] & Tor_2(M,Z) \arrow[dll] & \\
Tor_1(M,X) \arrow[r] & Tor_1(M,Y) \arrow[r] & Tor_1(M,Z) \arrow[dll] & \\
M \otimes X \arrow[r] & M \otimes Y \arrow[r] & M \otimes Z \arrow[r] & 0
\end{tikzcd}$$

Sketch of construction: take a resolution $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ with F_i projective or free. and then $Tor_i(M,N) = H_i(N \otimes F_\bullet)$. By chasing double complexes, we get $Tor_i(M,N) \cong Tor_i(N,M)$.

Proposition. M is A−flat if and only if $Tor_1(M,N) = 0, \forall N \iff Tor_j(M,N) = 0 \forall j \in \mathbb{N}$

Corollary. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact , then M_1, M_3 flat implies M_2 flat. M_2, M_3 flat implies M_1 is flat.

By the exact sequence: $Tor_2(M_3, N) \to Tor_1(M_1, N) \to Tor_1(M_2, N) \to Tor_1(M_3, N)$, we have the two modules on both sides are 0, and one of the two modules in the middle is also 0, thus the remaining one has to be 0.

Proposition. M is finitely generated A-module, A is Noetherian, then M is flat $\iff$ M is locally free.

Definition. X a scheme, $\mathcal{E}$ a quasi-coherent sheaf on X. We say $\mathcal{E}$ is $\mathcal{O}_X-$flat if $\mathcal{E}(SpecA)$ is $A-$flat for open affines $SpecA$. It is enough to check on open covers and furthermore, on stalks when X is Noetherian.

Remark. $\mathcal{E}$ is coherent, then it is $\mathcal{O}_X-$flat $\iff$ $\mathcal{E}$ is locally free $\iff$ $\mathcal{E}$ is a sheaf of sections of a vector bundle $E \to X$.

Definition. $\pi : Y \to X$ a map of schemes is flat if $\forall y \in Y$, $\mathcal{O}_{Y,y}$ is $\mathcal{O}_{X,\pi(y)}-$flat. Equivalently, $\forall y \in Y$ there exists $\mathrm{Spec}A \ni \pi(y)$, open in X and $\mathrm{Spec}B \subseteq \pi^{-1}(\mathrm{Spec}A)$ open in Y, such that B is flat over A.

For example, any open inclusion is flat since localization is flat. $U \hookrightarrow X$ also it is not necessarily finite. Since $\otimes$ is associative, we have composition of flat maps is flat.

April 13 – Gröbner Degeneration. Suppose we have a subscheme $Z_1 \subset X$ and want to find interesting degenerations. Given an action of the multiplicative group $\mathbb{G}_m = \mathrm{Spec}\, k[t, t^{-1}]$ or the additive group $\mathbb{G}_a = \mathrm{Spec}\, k[t]$ on X, we can compactify $\mathbb{G}_m$ to $B = \mathbb{A}^1$ or $\mathbb{G}_a$ to $B = \mathbb{P}^1$. Take $Z \subset X \times \mathbb{G}_m$ (or $X \times \mathbb{G}_a$) to be $\{(z, g) : g * z \in Z_1\}$. Take the closure in $X \times B$ and take the fiber over 0.

Ex: $X = \mathbb{A}^2$. Suppose $\mathbb{G}_m$ acts by $t \cdot (x, y) = (tx, ty)$. If $Z_1 \subset \mathbb{A}^2 = \langle xy - 1\rangle$, then $Z \subset \mathbb{A}^2 \times \mathbb{G}_m = \{(x, y, t) : (tx)(ty) - 1 = 0\}$. The fiber over 0 is $Z_0 = \emptyset$.
If instead we took the action $t \cdot (x, y) = (t^{-1}x, t^{-1}y)$, then $Z = \{(x, y, t) \in \mathbb{A}^2 \times \mathbb{G}_m : (t^{-1}x)(t^{-1}y) - 1 = 0\}$, so that $Z_0 = \{xy = 0\}$.

For $H \subset Aut(X)$, if G, H commute and Z_1 is H-invariant, then Z_0 is H-invariant. Z_0 is also G-invariant, where G acts on Z by $g \cdot (g_1, x) = (g_1 g^{-1}, gx)$.

Special case: For $X = \mathbb{A}^n, G = \mathbb{G}_m$ acts diagonally by $t \cdot \begin{bmatrix} x_1 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} t^{w_1}x_1 \\ \cdots \\ t^{w_n}x_n \end{bmatrix}$.
We get a degeneration from $Z_1 \subseteq \mathbb{A}^n$ to $Z_0 \subseteq \mathbb{A}^n$ with invariance by this 1-parameter subgroup of $\mathbb{G}_m^n$. Every closed subscheme of $\mathbb{A}^n$ has a flat degeneration to a $\mathbb{G}_m^n$-invariant subscheme of $\mathbb{A}^n$.

Claim: I is torus-invariant if and only if I is monomial.
Proof: Clearly, if I is monomial then it is torus invariant.
Suppose I is torus-invariant. Let M be the set of monomials in I. We want to show $\langle M\rangle = I$. Given $f \in I$, write $f = \sum_{A \in E} f_A x^A$. Then $\sum_{A \in E} f_A t_1^{A_1} \cdots t_n^{A_n} x^A \in I$, so $x^A \in I$ for $A \in E$, and so $f \in \langle M\rangle$.

April 15 – The semicontinuity theorem. We will show that cohomology groups "jump up" in flat families.

The question we want to answer is the following:

Question Let B be a base, and X be a proper and flat over B, and let $\pi : X \to B$. As we approach some $b_0 \in B$, how do the fibers $\pi^{-1}(b)$ change?

Theorem. Let B is noetherian, X is projective over B (i.e., closed subscheme of $\mathbb{P}^r_B$) and $\mathcal{E}$ be coherent sheaf of X. The the function $b \mapsto \dim_{k(b)} H^q(\pi^{-1}(b), \mathcal{E}|_{\pi^{-1}(b)})$ is upper semi-continuous: that is, it only "jumps upward" (if it does).

The proof we will present (next class) will be very close to R. Vakil's proof from his "Rising sea" chapter 28. One can also find the theorem in Hartshorne III.12.

This is only one of many theorems which say how things change as we specialize in flat families. Here are some other examples: **Facts**

i Number of connected components of $\pi^{-1}(b)$ is lower semicontinuous over a normal base
ii $\{b : \pi^{-1}(b) \text{ is singular}\}$ is closed
iii Rule of thumb: As you go more towards special fibers, things become more singular and more combinatorial.

Example. *"Jumping up" actually can happen* Take a twisted cubic X in $\mathbb{P}^3$, and degenerate it to a nodal cubic X' in a plane with nonreduced point in the node. X' can also be considered as a degeneration of X'', which is a disjoint union of a nodal cubic in a plane and a point outside of the plane. Then, we see that

$$H^0(X, \mathcal{O}) = k, H^0(X'') = k^2,$$

and we can check that $H^0(X') = k^2$: We have restriction map $H^0(X', \mathcal{O}) \twoheadrightarrow H^0(\alpha, \mathcal{O}) = k$, where α: nodal cubic. And the kernel of the surjection is k (take the defining equation of the hyperplane containing the curve near the singular point...)

Example. *Why do we need flatness* In $\mathbb{P}^1 \times \mathbb{A}^1$, and look at

$$\{((x_0 : x_1), t) : t(x_0 - x_1)(x_0 + x_1)\}$$

When $t \neq 0$, fiber is two points, so $H^0 = k^2$ and when $t = 0$, fiber is $\mathbb{P}^1$, so $H^0 = k$.

Towards the proof of the main theorem Our question is local on B, so say $B = \operatorname{Spec} A$. $H^q(X, \mathcal{E})$ is an A-module, and further it is even a finitely generated A-module (we showed this when A: field, and same argument works for A: Noetherian).

We see that

$$\mathfrak{p} \mapsto \dim_{A/\mathfrak{p}} H^q(X, \mathcal{E})/\mathfrak{p}H^q(X, \mathcal{E})$$

is upper semicontinuous. However, sadly, $\dim_{A/\mathfrak{p}} H^q(X, \mathcal{E})/\mathfrak{p}H^q(X, \mathcal{E}) \ncong H^q(\pi^{-1}(\mathfrak{p}, \mathcal{E}|_{\pi^{-1}(\mathfrak{p})})$ (see cohomology and base change theorem).

What does work, however, is Mumford's *cool lemma*!

Before we introduce Mumford's lemma, let's recall some results from commutative algebra.

Lemma. Let A be a commutative ring, and $X^\bullet$ be an exact complex of A-modules. If M: flat A-module, then $M \otimes X^\bullet$ is also exact.

Proof. This follows because a functor $M \otimes -$ takes a SES to a SES when M is flat. See Atiyah-Macdonald Proposition 2.19. (Our definition of flat module was that $M \otimes -$ sends SES to a SES. AM's definition of flat module is that tensoring with the module transforms all exact sequences into exact sequences. The two notions are equivalent due to proposition 2.19). □

Lemma. Again, let A be a commutative ring and $X^\bullet$, $Y^\bullet$ be complexes of A-modules. If $X^\bullet \to Y^\bullet$ is a quasi-isomorphism, and M is a flat A-module, then $X^\bullet \otimes M \to Y^\bullet \otimes M$ is a quasi-isomorphism.

Proof. Remember that $X^\bullet \to Y^\bullet$ is a quasi-isomorphism iff the mapping cone $C^\bullet$ is exact (See Weibel Corollary 1.5.4). Thus, using the previous lemma, we get

$X^\bullet \to Y^\bullet$ is a quasi-isomorphism $\iff$ $C^\bullet$ is exact $\iff$ $C^\bullet \otimes M$ is exact $\iff$ $X^\bullet \otimes M \to Y^\bullet \otimes M$ is a quasi-isomorphism.

□

Now we present Mumford's cool lemma

Lemma. (Mumford) Let A: commutative Noetherian ring. Let $C^\bullet$ be a complex of A-modules such that $C^n = 0$ for $n >> 0$ and $H^q(C^\bullet)$ be finitely generated A-module. Then there is a complex $K^\bullet$ of finitely generated A-modules and a map $K^\bullet \to C^\bullet$ which is a quasi-isomorphism.

This means that as long as $H^q(C^\bullet)$ is f.g, we can replace $C^\bullet$ by a complex $K^\bullet$ of finitely generated A-modules. We will present the proof of the lemma in next class.

April 17 - The Semicontiuity Theorem concluded. Today's goal is to go through the proof of the semicontinuity theorem. This approach is also discussed in Chapter 28 of Vakil's book.

Lemma (Mumford)**.** Suppose A is a noetherian ring, $C^\bullet$ is a complex of A-module such that $C^n = 0$ for $n \gg 0$, and also $H^q(C^\bullet)$ finitely generated. Then there exists a complex $K^\bullet$ of finitely generated free A-modules with a quasi-isomorphism $K^\bullet \to C^\bullet$.

Proof sketch. Suppose we have inductively built the diagram as below:

$$\begin{tikzcd}
 & & K^i \arrow[r, "\delta^i"] \arrow[d] & K^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{i+1} \arrow[r] & C^i \arrow[r] & C^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $H^q(K^\bullet) \to H^q(C^\bullet)$ for $q \geqslant i+1$ and $\ker(\delta^i)$ surjects onto $H^i(C^\bullet)$. Now we choose a surjection $p : A^N \twoheadrightarrow \ker(\ker(\delta^i) \twoheadrightarrow H^i(C^\bullet))$ and lift the image of each generator of A^N in C^i to C^{i+1}, as each generator of A^N maps down to 0 in $H^i(C^\bullet)$. So we get a map $\sigma : A^N \to C^{i-1}$ that makes the following diagram commutes:

$$\begin{tikzcd}
 & A^N \arrow[r, "p_N"] \arrow[d, "\sigma"] & K^i \arrow[r, "\delta^i"] \arrow[d] & K^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{i+1} \arrow[r] & C^i \arrow[r] & C^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Lastly, we choose a surjection $A^M \twoheadrightarrow H^{i-1}(C^\bullet)$, and lift this map to $p_M : A^M \to C^{i-1}$, so we get the following commutative diagram:

$$\begin{tikzcd}
 & A^M \oplus A^N \arrow[r, "{(0,p_N)}"] \arrow[d, "{(p_M,\sigma)}"] & K^i \arrow[r, "\delta^i"] \arrow[d] & K^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C^{i+1} \arrow[r] & C^i \arrow[r] & C^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Now this gives a quasi-isomorphism between $K^\bullet$ and $C^\bullet$, hence we are done. $\square$

We also note there is a variant: If C^i are flat and $C^i = 0$ for $i < 0$, then we may take $K^0, \cdots, K^n$ as finitely generated free modules, K^{-1} as finitely generated flat module (locally free), and $K^{-1} = 0$ for $i < -1$.

Theorem (Semicontinuity Theorem)**.** B is a noetherian scheme, $\mathcal{E}$ is a coherent sheaf on $\mathbb{P}^r_B$, and $\mathcal{E}$ is $\mathcal{O}_B$ flat. Then $\dim H^q$ us an upper semicontinuous function on B.

Proof. This statment is local on B, so we can take $B = \operatorname{Spec} A$. Now we have a Čech complex $C^\bullet$ on $\mathbb{P}^r_B$ as below:

$$\bigoplus \mathcal{E}(U_i) \to \bigoplus \mathcal{E}(U_{ij}) \to \cdots$$

which is a complex of flat A-modules. For any $\mathfrak{p} \in \operatorname{Spec} A$, H^q of $\mathcal{E}$ on fiber over $\mathfrak{p}$ is computed by $H^q(C^\bullet \otimes_A \operatorname{Frac}(A/\mathfrak{p}))$. By Mumford's lemma, we may build the complex $K^\bullet$ such that $K^\bullet \to C^\bullet$ is a quasi-isomorphism. Since all K^i and all C^i are flat, we have $K^\bullet \otimes_A \operatorname{Frac}(A/\mathfrak{p}) \to C^\bullet \otimes_A \operatorname{Frac}(A/\mathfrak{p})$ is still a quasi-isomorphism. Explicitly, as $K^i \cong A^{b_i}$ for some b_i, the map $K^i \xrightarrow{\delta^i} K^{i+1}$ is given by some $b_{i+1} \times b_i$ matrix with entries in A. Now we want to show $\dim_{k(\mathfrak{p})} H^q(K^\bullet \otimes_A k(\mathfrak{p}))$ is upper semicontinuous. Since

$$\begin{aligned}
\dim H^q &= \dim ker(\delta^q) - \dim \mathcal{I}m(\delta^{q-1}) \\
&= \dim K^q - \operatorname{Rank}(\delta^q) - \operatorname{Rank}(\delta^{q-1}) \\
&= \text{constant} - \text{(l.s.c.)} - \text{(l.s.c.)},
\end{aligned}$$

our conclusion hence follows. $\square$