SYMMETRIC FUNCTION CHEAT SHEET

This cheat sheet concerns transition functions and inner products between the m, e, h and s bases for the ring of symmetric functions. The results were either proved in class/problem sets, or are easy consequences of such results. At some point, I may add in facts about the p's.

Transition matrices

From h to m. We have $h_\lambda = \sum_\mu A_{\lambda\mu} m_\mu$, where $A_{\lambda\mu}$ is the number of nonnegative integer matrices with row sum λ and column sum μ. Note that $A_{\lambda\mu} = A_{\mu\lambda}$.

From e to m. We have $e_\lambda = \sum_\mu B_{\lambda\mu} m_\mu$, , where $B_{\lambda\mu}$ is the number of $(0,1)$ matrices with row sum λ and column sum μ. Note that $B_{\lambda\mu} = B_{\mu\lambda}$. We have $B_{\lambda\lambda^T} = 1$ and $B_{\lambda\mu} = 0$ if $\mu \not\leq \lambda^T$.

From h to s, s to m and e to m. We have $h_\mu = \sum_\lambda K_{\lambda\mu} s_\mu$, $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ and $e_\mu = \sum_\lambda K_{\lambda^T\mu} s_\lambda$. Here $K_{\lambda\mu}$ is the ***Kostka number***; it is the number of SSYT of shape λ and content μ. We have $K_{\lambda\lambda} = 1$ and $K_{\lambda\mu} = 0$ if $\mu \not\leq \lambda$.

From m to s, s to h and s to e. We have $s_\mu = \sum_\lambda L_{\lambda\mu} h_\lambda$, $s_\mu = \sum_\lambda L_{\lambda\mu^T} e_\lambda$, $m_\lambda = \sum_\mu L_{\lambda\mu} s_\mu$. Here $L_{\lambda\mu}$ is the ***inverse Kostka number***. Letting $\rho = (n-1, n-2, \ldots, 2, 1, 0)$, it is computed by taking the sum of $(-1)^v$ over all v such that $v(\mu) + v(\rho) - \rho$ is a permutation of λ or, equivalently, $\mu + \rho - v(\rho)$ is a permutation of λ. It can also be described as the number of rim hook tableaux of shape μ and weight λ, counted with sign. The $s \rightsquigarrow h$ and $s \rightsquigarrow e$ identities are called ***Jacobi-Trudi*** and ***dual Jacobi-Trudy***. The fact that the $s \rightsquigarrow m$ and $m \rightsquigarrow s$ matrices are inverse is equivalent to the ***ratio of alternants formula***. The inverse Kostka number is sometimes denoted $K^{-1}_{\lambda\mu}$.

From e to h. Define $D_{\lambda\mu}$ by $h_\lambda = \sum_\mu D_{\lambda\mu} e_\mu$. Then we also have $e_\lambda = \sum_\mu D_{\lambda\mu} h_\mu$. We have $D_{\lambda\lambda} = \pm 1$ and $D_{\lambda\mu} = 0$ if $\mu \not\leq \lambda$. I do not know a reasonable combinatorial formula for $D_{\lambda\mu}$.

Dual bases

The inner product $\langle\ ,\ \rangle$ is symmetric and positive definite. We have

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} \quad \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

In other words, h and m are dual bases; s is self dual. The dual to the e's are sometimes called the "forgotten symmetric functions" and denoted f_λ; they don't seem to have any useful characterization other than being dual to the e's. Other inner products between symmetric functions can usually be extracted from the above dualities and the expansions in the previous section.

The Cauchy products

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda h_\lambda(x) m_\lambda(y) = \sum_{\lambda\mu} A_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_\lambda s_\lambda(x) s_\lambda(y)$$
$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda e_\lambda(x) m_\lambda(y) = \sum_{\lambda\mu} B_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_\lambda s_{\lambda^T}(x) s_\lambda(y).$$

The involution ω

We have $\omega(e_\lambda) = h_\lambda$, $\omega(h_\lambda) = e_\lambda$ and $\omega(s_\lambda) = s_{\lambda^T}$. We have $\omega^2 = \mathrm{Id}$.

Standardness of notations

The notations m, e, h and s are completely standard. If the reader already knows that you are writing about symmetric functions, probably no definition is needed at all; otherwise, a quick "s_λ is the Schur polynomial" should do. Every combinatorialist agrees on what $K_{\lambda\mu}$ means, but I probably would write "$K_{\lambda\mu}$ is the Kostka number". The notations $A_{\lambda\mu}$ and $B_{\lambda\mu}$ are reasonably common, but not enough to use them without definition. All the others are local to this course.