GL_n REPRESENTATION THEORY NOTES FOR 12-03

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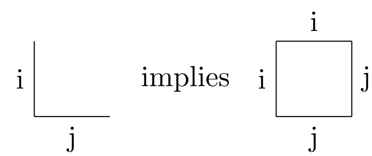
As with the last lecture, this one is based on John Stembridge' s paper: *A local characterization of simply-laced crystals*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 4807–4823.

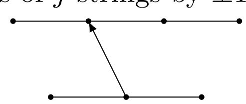
1. Recall

Last class, we defined ***regular crystals***. These are crystals with the property that

- If $|i-j| \geq 2$, then we can "fill in squares":



- If $|i-j| = 1$, then i changes lengths of j-strings by ± 1, and vice versa:



- and a few other specific rules (see the previous lecture).

2. Theorem for today

Today's main results are the following two theorems:

Theorem 1 (Stembridge)**.** *A finite connected regular crystal has unique high weight element.*

Theorem 2 (Stembridge)**.** *If B and B' are finite connected regular crystals with high weight elements u and u', and if $\mathrm{wt}(u) = \mathrm{wt}(u')$, then $B \cong B'$.*

All the hard work takes place in the following lemma:

Lemma 1. *Let B be a regular crystal, u a high weight vector, and $v \in B$ with*

$$v = e_k f_{i_r} \cdots f_{i_2} f_{i_1} u, \text{ for some } k, i_1, i_2, \ldots, i_r \in \{1, \ldots, n-1\}.$$

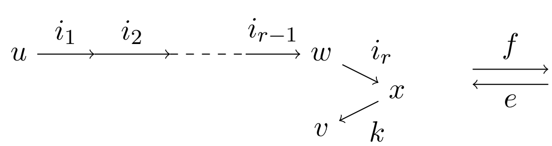
Then in fact

$$v = f_{j_{r-1}} \cdots f_{j_1} u \text{ for some } j_1, \ldots, j_{r-1} \in \{1, \ldots, n-1\}.$$

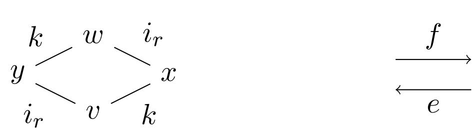
In other words, it's possible to 'omit' the upwards step e_k, and get from u to v just using 'downwards steps' f_{i_ℓ}.

Proof. By induction on r. If $r = 0$, $e_k u = 0$ since u is high weight, and $0 \notin B$, this is a contradiction.

For $r > 0$, let's say we have... (rest of proof is in the diagram)



There are three cases. First of all, if $i_r = k$, then $v = w$, so we're done. Second, if $|i_r - k| \geq 2$, then we can "fill in the square" to get:



So, in other words, we have

$$u \xrightarrow{i_1} \xrightarrow{i_2} - - - - \xrightarrow{i_{r-1}} w, \quad y \xrightarrow{k} w$$

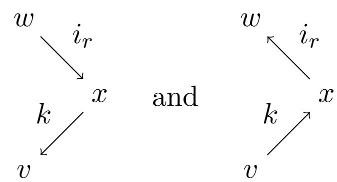
so by induction,

$$u \xrightarrow{j_1} \xrightarrow{j_2} - - - - \xrightarrow{j_{r-2}} y \xrightarrow{i_r} v.$$

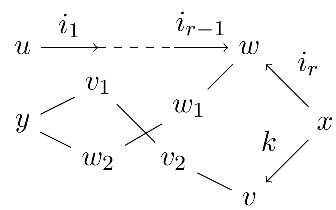
In the third case, $|i_r - k| = 1$. This splits into three sub-cases: first,

$$w \xrightarrow{i_r} x \xleftarrow{k} v \quad \text{is a contradiction.}$$

Next, to deal with



we use the "filling in squares" axiom and continue as in case 2. Finally, if both arrows point out of x, we have to use the 'diamond-shaped' axiom to pull back two steps:



Inductively, we can pull back three times in a row: we get a line of arrows from u to w_1; then from u to w_2; then from u to y. Finally, we fill in the arrows

$$u \longrightarrow - - - - \longrightarrow y \text{ --- } v_1 \text{ --- } v_2 \text{ --- } v.$$

This completes the proof. □

Corollary 1. *If B is regular, u is high weight, v in the same connected component as u, then*

$$v = f_{j_s} \cdots f_{j_1} u.$$

Proof. Write v in terms of u using both e's and f's, then eliminate e's using the lemma. □

Question: How should we think about this proof? Here are some scattered thoughts, added by David.

One should compare this to the proof that every $\mathfrak{gl}_n$ irrep V has a unique high weight vector u. The key lemma was that, for u high weight, the vectors $f_{i_r} \cdots f_{i_2} f_{i_1} u$ span V. To show this, it was enough to show that $e_k f_{i_r} \cdots f_{i_2} f_{i_1} u$ is in the span of vectors of the form $f_{j_{r-1}} \cdots f_{j_1} u$. See the November 21 notes for a very similar argument in the Lie algebra setting.

A better way, perhaps, to understand the argument of Nov. 21 is through the PBW theorem. The (easy part of the) PBW theorem shows that $\mathcal{U}(\mathfrak{gl}_n)$ is spanned by monomials of the form $\left(\prod_{i>j} E_{ij}^{a_{ij}}\right)\left(\prod_d E_{kk}^{b_k}\right)\left(\prod_{i<j} E_{ij}^{c_{ij}}\right)$. From this, we conclude that V is spanned by vectors of the form $\left(\prod_{i>j} E_{ij}^{a_{ij}}\right)\left(\prod_d E_{kk}^{b_k}\right)\left(\prod_{i<j} E_{ij}^{c_{ij}}\right) u$. But such a vector is 0 unless all the c_{ij} are 0 (since u is high weight) and in that case is proportional to $\left(\prod_{i>j} E_{ij}^{a_{ij}}\right) u$ (since u is a weight vector).

If we had something like the universal enveloping algebra for crystals, we could imagine showing that this algebra was spanned by monomials where we do all of the e's first, and then all of the f's. Since any e annihilates u, and we know that we can get to anywhere in B by applying e's and f's to u, this shows that we can get anywhere from u by doing the e's first. I am not familiar with a universal-enveloping-algebra like object for crystals, but in the appendix I give a proof like this.

3. Proofs of Theorems 1 and 2

Now we can prove Theorem 1 above (uniqueness of high-weight vectors in finite connected regular crystals).

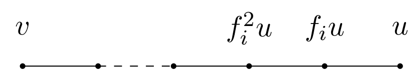
Proof of Theorem 1. First, any finite crystal has a high weight element (apply e operators, which increase the weight, until we can't anymore). Suppose u and $v \neq u$ are both high weight. By the above corollary,

$$v = f_{j_s} \cdots f_{j_1} u.$$

Then $e_{j_s} v \neq 0$, a contradiction. □

Remark. This is exactly like the corresponding proof for $\mathfrak{gl}_n$, namely: if you have a high weight vector, and you know you can reach any other vector by (only) going downwards, then there's no room for any other high weight vector.

An important observation is the following: if B is a crystal and u is high weight, then its weight vector is decreasing, that is, $\mathrm{wt}(u) = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$. To see this, consider the i-string through u:



Then we see that

$$\mathrm{wt}(v) = s_i \,\mathrm{wt}(u) = (\lambda_1, \lambda_2, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n),$$

and so

$$\mathrm{wt}(v) = \mathrm{wt}(u) + \phi \cdot (0, 0, \ldots, -1, 1, \ldots, 0),$$

so $\lambda_{i+1} = \lambda_i - \phi$. Thus $\lambda_{i+1} \leq \lambda_i$. We will use this fact in our proof of the second theorem.

Proof of Theorem 2. Let $\mathrm{wt}(u) = \mathrm{wt}(u') = (\lambda_1, \ldots, \lambda_n)$ be the common high-weight vectors of the two crystals B and B'. Define height functions $h : B \to \mathbb{Z}$ and $B' \to \mathbb{Z}$ as follows: if $\mathrm{wt}(b) = (\kappa_1, \ldots, \kappa_n)$, then

$$h(b) = \sum_{i=1}^{n} (\kappa_i - \lambda_i) i.$$

In particular,

$$h(f_i b) = h(b) + 1 \text{ if } f_i b \neq 0,$$

and similarly

$$h(e_i b) = h(b) - 1 \text{ if } e_i b \neq 0.$$

We show, by induction on r, that there is a bijection between

$$B_{\leq r} := \{b \in B, h(b) \leq r\}$$

and the corresponding set $B'_{\leq r}$, such that

- The bijection commutes with e_i and f_i as maps **within** $B_{\leq r} \sqcup \{0\}$ and $B'_{\leq r} \sqcup \{0\}$,
- The bijection preserves the quantities $\varepsilon_i(b)$ and $\phi_i(b)$ (which measure the distance from b to the ends of the i-string through b.

Since our crystals are finite, when r is large enough, this gives a bijection $B \leftrightarrow B'$ commuting with e_i, f_i. (David said: finiteness is not as important as I'm making it sound – the paper this proof comes from doesn't always require it. But we're assuming finiteness for simplicity.)

The base case is $r = 0$. In this case $B_{\leq 0} = \{u\}$ and $B'_{\leq 0} = \{u\}$: the map is obviously bijective, and it "agrees" with the e_i's, since $e_i u = e'_i u' = 0$ for any i. The maps f_i all map **out of** these sets, so we don't need to verify anything else. This shows that the first property holds.

For the second property, note that u is at the e-end of its i-string, so $\varepsilon_i(u) = 0$ and $\phi_i(u) = \lambda_i - \lambda_{i+1}$. The same fact holds for u'.

At this point, the rest of the proof is essentially an exercise: it suffices to show that the f arrows act appropriately (since any point can be reached just using f arrows, by the Lemma); and then it's just a matter of considering the various ways f arrows can come together, and applying the various axioms of regular crystals.

Proof continued on 12/05. For the inductive case, suppose we have $\alpha : B_{\leq r} \to B'_{\leq r}$. We wish to extend this bijection to $r+1$.

We know that every $b \in B_{r+1}$ is $f_i c$ for some i and some $c \in B_r$. First of all, we observe that

$$f_i c = 0 \text{ if and only if } f_i \alpha(c) = 0.$$

To see this, note that the first statement is equivalent to $\phi_i(c) = 0$, and similarly for the second. (The bijection α preserves ϕ_i by induction). In other words, α induces a bijection between

$$\{(c, i) : c \in B_r, i \in \{1, \dots, n\}, f_i c \neq 0\}$$

and the corresponding set for B'_r. To construct α on B_r, we show that

$$f_i(c_1) = f_j(c_2) \text{ if and only if } f_i(\alpha(c_1)) = f_j(\alpha(c_2)), \text{ for all } c_1, c_2 \in B_r, 1 \leq i, j \leq n-1.$$

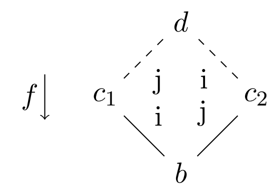
(We only need to prove the forward direction, since the situation is symmetric between B_r and B'_r.) We'll also need to check that the new α still preserves ε_i, ϕ_i.

If $i = j$, then

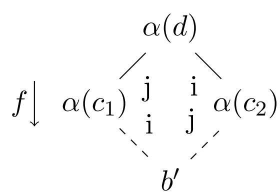
$$f_i c_1 = f_i c_2 \text{ if and only if } c_1 = c_2,$$

so the claim follows immediately.

Next, if $|i - j| \geq 2$, then we have the picture

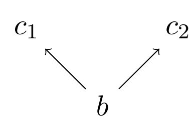


(Read this like a commutative diagram asserting an 'existence' statement: assuming the existence of the solid lines, the regularity axioms imply that we can fill in the dashed lines.) Apply α to the picture: by induction, we get



This shows that $f_i \alpha(c_1) = f_j \alpha(c_2) = b'$.

Finally, if $|i - j| = 1$, there are four cases (based on the orientations of the edges). The most interesting case is



Here we use the last regularity axiom (governing the 'diamond-shaped' arrow setup) to pull back two steps; then apply α throughout. Finally, we use regularity again to find a b' mapping to $\alpha(c_1), \alpha(c_2)$.

Now we've defined α everywhere; we need to show it still preserves ε_i and ϕ_i. So let $b = f_i c$ as earlier; we want to consider $\varepsilon_i(b), \phi_i(b)$.

If $i = j$,

$$\phi_i(c) = \phi_i(b) - 1 \text{ and } \varepsilon_i(c) = \varepsilon_i(b) + 1.$$

If $|i-j| \geq 2$,

$$\phi_i(c) = \phi_i(b) \text{ and } \varepsilon_i(c) = \varepsilon_i(b).$$

As usual, the case $|i-j| = 1$ is the interesting one. For simplicity, take $j = i+1$. First of all, we know α commutes with e_i, so in particular,

$$e_i(b) = 0 \text{ if and only if } e_i(\alpha(b)) = 0.$$

Note that in this case $\varepsilon_i(b) = \varepsilon_i(\alpha(b)) = 0$, and we can compute $\phi_i(b)$ from

$$\begin{aligned}\mathrm{wt}(b) &= \mathrm{wt}(c) + (0, \ldots, 0, \underbrace{-1}_{i+1}, \underbrace{1}_{i+2}, 0, \ldots, 0) \\ &= \mathrm{wt}(\alpha(c)) + (0, \ldots, 0, \underbrace{-1}_{i+1}, \underbrace{1}_{i+2}, 0, \ldots, 0).\end{aligned}$$

So ϕ_i is preserved.

Otherwise $e_i(b) \neq 0$, we break into some more cases. In each case, the regularity axioms show that the orientations of the f_{i+1} edges b–c_2 and $\alpha(b)$–$\alpha(c_2)$ match. Then we can compute $\varepsilon_i(b), \phi_i(b)$ from the knowledge of $\varepsilon_i(c_2), \phi_i(c_2)$ and this orientation. □

Appendix: An alternate route to Corollary 1 (Added by David)

I don't know whether the following proof of Corollary 1 will seem more or less clear, but I present it as an alternate perspective. Let B be a finite connected regular crystal, and let u and v be elements of B. Consider all ways to write $v = g_r g_{r-1} \cdots g_2 g_1 u$ for some sequence of crystal operators $g_r \cdots g_1$. We assign a score to each letter g_j. If g_j is an f, the score is 0. If g_j is an e, then the score is $2^{h(g_{j-1}g_{j-2}\cdots g_1 u)}$ where h is the height function (so f's increase height, e's decrease height and height is always ≥ 0). The score of the expression $g_r \cdots g_2 g_1 u$ is the sum of the scores of each letter.

Suppose that the word $g_\bullet$ contains $\cdots e_i f_j \cdots$ at some point. We can now make the following replacements:

- If $i = j$, we can delete the pair $e_i f_i$. This removes one letter (that e_i) which contributes positively to the score and maintains the score of every other letter; hence it decreases the score.
- If $|i-j| \geq 2$, we can replace $e_i f_j$ by $f_j e_i$. This changes the score of that e_i from 2^h to 2^{h-1} (for some h) and keeps all other scores the same.
- If $|i-j| = 1$ and the edges corresponding to these letters are oriented as $\xrightarrow{e_i}\xrightarrow{f_j}$ or $\xleftarrow{e_i}\xleftarrow{f_j}$, then we may replace $e_i f_j$ by $f_j e_i$. As above, this changes the contribution of that e_i from 2^h to 2^{h-1}.
- If $|i-j| = 1$ and the corresponding edges are oriented as $\xleftarrow{e_i}\xrightarrow{f_j}$, then we may replace $e_i f_j$ by $f_j f_j f_i e_j e_i e_i$. This removes a letter with score 2^h and inserts new letters with scores $2^{h-1} + 2^{h-2} + 2^{h-3} = (7/8)2^h$.

Thus, whenever we have $e_i f_j$, we can decrease the score. But the score is a positive integer, so it cannot decrease forever. So, if we keep making the above replacements, we will eventually get to a word which is of the form $fff \cdots fe \cdots eee$.

In particular, if u is a hight weight element, we will eventually get to a formula $v = f_{j_r} \cdots f_{j_2} f_{j_1} u$, since u is annihilated by every e_i.