## NOTES FOR DECEMBER 5

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### 1. Uniqueness of Connected Regular Crystals

Last time, we stated the following theorem. If $B$ and $B'$ are finite, connected regular crystals with high weight elements $u$ and $u'$ such that $\mathrm{wt}(u) = \mathrm{wt}(u') = \lambda$, then $B \cong B'$. Let's prove this.

For $b \in B \cup B'$ with $\mathrm{wt}(b) = (k_1, \ldots, k_n)$, let $\mathrm{ht}(b) = \sum_{i=1}^{n} i(k_i - \lambda_i)$. Then $\mathrm{ht}(e_i b) = \mathrm{ht}(b) - 1$, and $\mathrm{ht}(f_i b) = \mathrm{ht}(b) + 1$.

Define $B_r = \{b \in B \mid \mathrm{ht}(b) = r\}$ and $B'_r = \{b' \in B' \mid \mathrm{ht}(b') = r\}$. Set $B_{\le r} = \{b \in B \mid \mathrm{ht}(b) \le r\}$ and $B'_{\le r} = \{b' \in B' \mid \mathrm{ht}(b') \le r'\}$.

We will show by induction on $r$ that there is a bijection $\alpha : B_{\le r} \to B'_{\le r}$ such that

- $\alpha$ commutes with $e_i$ and $f_i$, whenever these map within $B_{\le r} \sqcup \{0\}$ and $B'_{\le r} \sqcup \{0\}$.
- $\alpha$ preserves $\epsilon_i$ and $\phi_i$, where

$$\epsilon_i(b) = \max\{k \mid e_i^k(b) \neq 0\}$$

$$\phi_i(b) = \max\{k \mid f_i^k(b) \neq 0\}$$

For the base case, let $r = 0$. $B_{\le 0} = \{u\}$, $B'_{\le 0} = \{u'\}$. We have $\epsilon_i(u) = \epsilon_i(u') = 0$, and $\phi_i(u) = \phi_i(u') = \lambda_i - \lambda_{i+1}$.

For the inductive step, say we have $\alpha : B_{\le r} \to B'_{\le r}$. We need to construct

$$\alpha : B_{r+1} \to B'_{r+1}.$$

Each $b \in B_{r+1}$ is $f_i c$ for some $i$ and some $c \in B_r$. We claim that for $i \in \{1, 2, \ldots, n-1\}$, and $c \in B_r$, we have $f_i c = 0 \Leftrightarrow f_i \alpha(c) = 0$.

To prove this, note that we have

$$f_i(c) = 0 \Leftrightarrow \phi_i(c) = 0 \Leftrightarrow \phi_i(\alpha(c)) = 0 \Leftrightarrow f_i \alpha(c) = 0$$

So $\alpha$ induces a bijection between

$$\{(c, i) \mid c \in B_r, i \in \{1, 2, \ldots, n-1\}, f_i(c) \neq 0\}$$

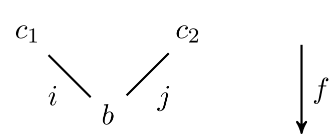
and

$$\{(c', i) \mid c' \in B'_r, i \in \{1, 2, \ldots, n-1\}, f_i(c') \neq 0\}$$

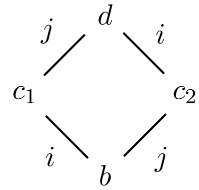
To construct $\alpha$ on $B_r$, we must show that $f_i(c_1) = f_j(c_2) \Leftrightarrow f_i(\alpha(c_1)) = f_j(\alpha(c_2))$ for $c_1, c_2 \in B_r$, $1 \le i, j \le n-1$. We will also need to check that $\alpha$ preserves $\epsilon_i$ and $\phi_i$. It suffices to show the forward direction, since the problem is symmetric. Let's take cases.

Suppose $i = j$. Then $f_i(c_1) = f_i(c_2) \Leftrightarrow c_1 = c_2 \Rightarrow \alpha(c_1) = \alpha(c_2) \Leftrightarrow f_i(c_1) = f_i(c_2)$.

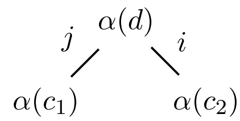
Next, suppose $|i - j| \ge 2$. If $f_i(c_1) = f_j(c_2) = b$, we have



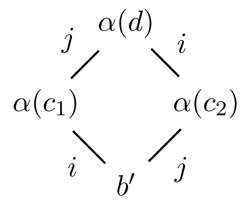

then by regularity we have



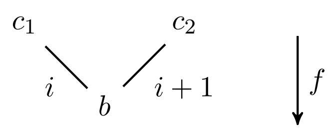
And hence, inductively, we have



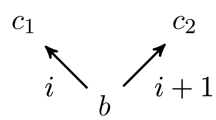

So using regularity again, we have



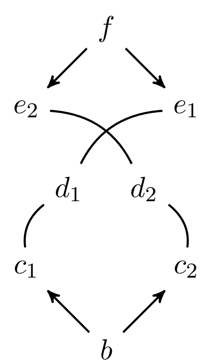

If $|i-j|=1$, we have



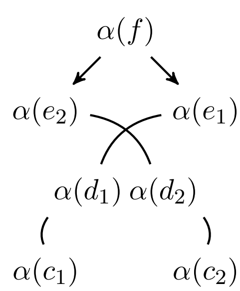

There are four cases, depending on the orientation of these edges. The most interesting case is




By regularity, we must have

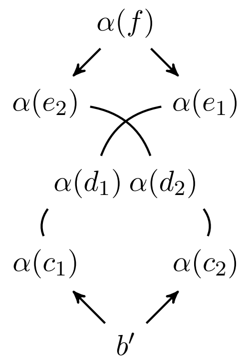



Inductively we have




We know the arrowheads on the top level of this diagram by the second part of the inductive hypothesis.

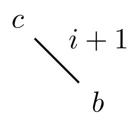
By regularity, this yields




We'll just sketch the rest of the proof.

Say $b = f_j(c)$. We want to consider $\epsilon_i(b)$ and $\phi_i(b)$. If $i = j$, then $\phi_i(c) = \phi_i(b) - 1$, $\epsilon_i(c) = \epsilon_i(b) + 1$. If $|i - j| \geq 2$, then $\phi_i(c) = \phi_i(b)$ and $\epsilon_i(c) = \epsilon_i(b)$.

Suppose $|i - j| = 1$. For simplicity say $j = i + 1$.



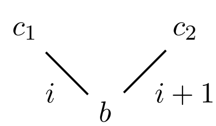

We claim $e_i(b) = 0 \Leftrightarrow e_i(\alpha(b)) = 0$. This follows since we already made $\alpha$ commute with $e_i$. If $e_i(b) = 0$ and $e_i(\alpha(b)) = 0$ then $\epsilon_i(b) = \epsilon_i(\alpha(b)) = 0$.

Note that everything we've done so far is weight-preserving. So we can compute $\phi_i(b)$ from

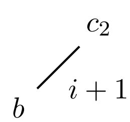
$$\begin{aligned} \mathrm{wt}(b) &= wt(c) + (0, \cdots, 0, -1, 1, \cdots, 0) \\ &= \mathrm{wt}(\alpha(c)) + (0, \cdots, 0, -1, 1, \ldots, 0) \end{aligned}$$

where the $-1$ occurs at position $i + 1$ and the 1 as position $i + 2$.

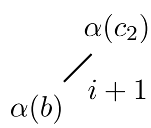
If $e_i(b) \neq 0$, set $c_1 = e_i(b)$, $c_2 = c$.




We get the same cases, and regularity axioms show in each case that the orientation of edges



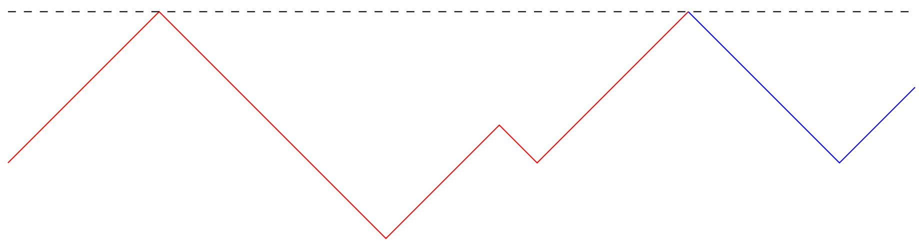

and




have to match up. We can compute $\epsilon_i(b)$ and $\phi_i(b)$ from $\epsilon_i(c_2)$, $\phi_i(c_2)$ and this orientation. This finishes the proof.

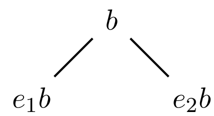
## 2. Word Crystals are Regular

It's easy to show that $(e_i, f_i)$ and $(e_j, f_j)$ commute for $|i - j| \geq 2$. Also, $(e_{i+1}, f_{i+1})$ changes the length of the $(e_i, f_i)$ strings by $\pm 1$. We know that $e_{i+1}$ turns an $i + 2$ into an $i + 1$. Thus it inserts an $i + 1$ into the $(i, i + 1)$ mountain range.



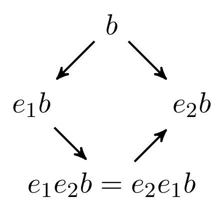
Inserting an $i+1$ in the region colored red is lengthing; inserting an $i+1$ into the blue region is shortening.

What's hard is figuring out the interaction of $(i, i+1)$ and $(i+1, i+2)$. We want to check the case

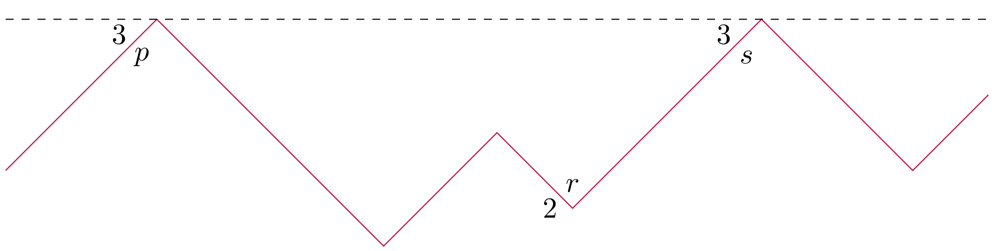



We may assume that $b = w_1 w_2 \cdots w_d$ where $w_j \in \{1, 2, 3\}$. Let $p$ such that $w_p = 3$ and $e_2$ changes $w_p$ from a 3 to a 2. Let $r$ be such that $w_r = 3$ and $e_1$ changes $w_r$ to a 1. We may assume that $p < r$.

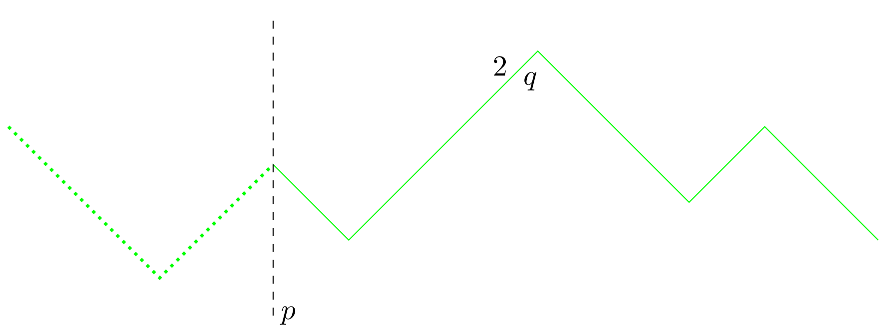
Let's look at the $(2,3)$ mountain range. We'll need to take cases again. After position $r$, does the mountain range re-acheive its maximum value of not? If there is no further maximum, then $e_1 e_2 b = e_2 e_1 b$, and both change $w_p$ from 2 to 2 and $w_r$ from 2 to 1.




The hard case happens when there is another maximum. In this case, let $s$ be the index for the 3 which occurs at the next maximum following $r$.




Now look at the $1,2$-range. Delete the $1,2$ string before position $p$, and recompute the lit and unlit sets. Let $q$ be the position of the rightmost let 2-edge which lies to the left of $r$.




(If there is no such edge, we are in yet another case.) So we have $p < q < r < s$. The only letters which change are in positions $p, q, r, s$.

(This material added by David.) We'll give an example and leave the general proof to the reader. Let our starting word be 332233.

We have
$$\begin{array}{cc} pqr\ \ s & \\ 332233 & \xrightarrow{e_1} \\ 332133 & \xrightarrow{e_2} \\ 332132 & \xrightarrow{e_2} \\ 322132 & \xrightarrow{e_1} \\ 321132 & \end{array}$$
and also
$$\begin{array}{cc} pqr\ \ s & \\ 332233 & \xrightarrow{e_2} \\ 322233 & \xrightarrow{e_1} \\ 322133 & \xrightarrow{e_1} \\ 321133 & \xrightarrow{e_2} \\ 321132 & \end{array}$$

The general pattern looks like this, except that there may be arbitrarily long strings between positions $p$, $q$, $r$ and $s$.