THE FROBENIUS CHARACTER MAP

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Let $d \leq n$. Then Schur-Weyl duality gives us an equivalence of categories:

$$\{\text{Finite dimensional } S_d \text{ representations}\} \to \{\text{Polynomial } GL_n \text{ representations where } t \cdot \mathrm{Id} \text{ acts by } t^d \}$$

Every element of the left hand category has a character, which is a class functton on S_d. Let Λ^d denote the vector space of symmetric polynomials of degree d. To every representation in the right hand category, taking the character assigns an element Λ^d. So we have a linear map

$$F : \{\text{Class functions on } S_d\} \to \Lambda^d.$$

This map is called the ***Frobenius character map***, and it is the goal of this note to describe it.

For a permutation $\sigma \in S_d$, let $c(\sigma)$ be the partition whose parts are the lengths of the cycles of σ. For example, the identity permutation maps to 1^d; a simple transposition maps to $2\ 1^{d-1}$.

1. The h, e and s bases

We can figure out some values of F by looking at the inverse correspondence when $n = d$.

1.1. The h basis.

We know that h_λ is the character of the representation $\bigotimes_k \mathrm{Sym}^{\lambda_k}(V)$, which I'll abbreviate H. I'll abbreviate the $(1,1,1,1,1,1,1)$ weight space by H_0.

For concreteness sake, take $\lambda = (4,2,1)$. Then H_0 has as basis elements of the form

$$(z_{i_1} z_{i_2} z_{i_3} z_{i_4}) \otimes (z_{j_1} z_{j_2}) \otimes z_k$$

where $\{1,2,3,4,5,6,7\}$ is the disjoint union of $\{i_1, i_2, i_3, i_4\}$, $\{j_1, j_2\}$ and $\{k\}$. So a basis for H_0 can be indexed by partitions of $[d]$ into sets of size λ_1, λ_2, $\ldots$, λ_r. The symmetric group acts by permuting those set partitions.

If a group G acts by permuting a finite set X, then the character of the permutation representation $\mathbb{C}X$ is

$$\chi_{\mathbb{C}X}(g) = \#(X^g)$$

In our case, we come to the following conclusion: Let $\mu = c(\sigma)$. Then $\chi_{H_0}(\sigma)$ is the number of ways to partition the multiset $\{\mu_1, \mu_2, \ldots, \mu_s\}$ into multisubsets whose sizes are $(\lambda_1, \lambda_2, \ldots, \lambda_r)$. For example, suppose that $\lambda = (4,2,1)$ and $c(\sigma) = (2,2,1,1,1)$. Then $\chi_{H_0}(\sigma) = 9$, corresponding to

$$(2+2, 1+1, 1) \quad (2+2, 1+1, 1) \quad (2+2, 1+1, 1) \quad (2+1+1, 2, 1) \quad (2+1+1, 2, 1)$$
$$(2+1+1, 2, 1) \quad (2+1+1, 2, 1) \quad (2+1+1, 2, 1) \quad (2+1+1, 2, 1)$$

The coloring is meant to make it clear that we must keep track of the distinct identities of the two 2's and the three 1's.

So the character of H_0 is the class function

$$\sigma \mapsto \#\{\text{set partitions of } c(\sigma) \text{ with parts of size } (\lambda_1, \lambda_2, \ldots, \lambda_r)\},$$

and F maps it to h_λ.

1.2. The e basis.

Similarly, we know that e_λ is the character of the representation $\bigotimes_k \bigwedge^{\lambda_k}(V)$. Let $E = \bigotimes_k \bigwedge^{\lambda_k}(V)$ and let E_0 be the $(1,1,\ldots,1)$ weight space. Taking $\lambda = (4,2,1)$ again, E_0 has as a basis the products of minors

$$\Delta_{i_1 i_2 i_3 i_4} \Delta_{j_1 j_2} \Delta_k.$$

Once again, our basis is indexed by set-partitions of $\{1,2,\ldots,n\}$ into sets of size $(\lambda_1, \lambda_2, \ldots, \lambda_r)$, but there is a sign factor. We deduce that

$$E_0 \cong H_0 \otimes \mathrm{sign}.$$

The character of E_0 is the class function

$$\sigma \mapsto (-1)^\sigma \cdot \#\{\text{set partitions of } c(\sigma) \text{ with parts of size } (\lambda_1, \lambda_2, \ldots, \lambda_r)\},$$

We deduce that this character maps to e_λ.

This would have perhaps been a slicker proof that ω corresponds to tensor with sign, since we know that $\omega(h_\lambda) = e_\lambda$.

1.3. **The s basis.** We defined the Specht module $Sp(\lambda)$ to be the $(1, 1, \ldots, 1)$ weight space of $V_\lambda(d)$. So the character of the Specht module maps to s_λ.

2. A general formula

Let $f : S_d \to \mathbb{C}$ be a class function. We claim that

$$F(f) = \frac{1}{d!} \sum_{\sigma \in S_d} f(\sigma) \operatorname{Tr}(\sigma \times \operatorname{diag}(t_1, t_2, \ldots, t_n)) \tag{1}$$

Here $\sigma \times \operatorname{diag}(t_1, t_2, \ldots, t_n))$ is an element of $S_d \times GL_n$, and we are considering its action on $V^{\otimes d}$ where $V = \mathbb{C}^n$.

It is enough to prove this result for f the character of an S_d-irrep. Let χ_λ be the character of $Sp(\lambda)$. Set

$$\pi_{Sp(\lambda)} = \frac{\dim Sp(\lambda)}{d!} \sum_{\sigma \in S_d} \chi_\lambda(\sigma) \rho_{Sp(\lambda)}(\sigma),$$

an element in $\mathbb{C}[S_d]$. From the October 3 lecture, $\pi_{Sp(\lambda)}$ acts by 1 on $Sp(\lambda)$ and acts by 0 on $Sp(\mu)$ for $\mu \neq \lambda$. We have

$$\frac{1}{d!} \sum_{\sigma \in S_d} \chi_{Sp(\lambda)}(\sigma) \sigma \times \operatorname{diag}(t_1, t_2, \ldots, t_n) = \frac{1}{\dim Sp(\lambda)} \pi_{Sp(\lambda)} \otimes \operatorname{diag}(t_1, t_2, \ldots, t_n).$$

By Schur-Weyl duality, the trace of the above operator on $V^{\otimes d}$ is $s_\lambda(t_1, t_2, \ldots, t_n)$, since it acts by 0 on $Sp(\mu) \otimes V_\mu(n)$ for $\mu \neq \lambda$ and acts by $\frac{1}{\dim Sp(\lambda)} \times \operatorname{diag}(t_1, \ldots, t_n)$ on $Sp(\lambda) \otimes V_\lambda(n)$. $\square$

3. The power symmetric functions

We can now see where the power symmetric functions come from. Let us extend F to functions on S_d which are not class functions, using formula (1). Let σ be an element of S_d with $c_\sigma = \mu$, and let δ_σ be the function which is 1 on σ and 0 elsewhere. So

$$F(\delta_\sigma) = \frac{1}{d!} \operatorname{Tr}\left(\sigma \times \operatorname{diag}(t_1, \ldots, t_n)\right).$$

Consider the action of $\sigma \times \operatorname{diag}(t_1, \ldots, t_n)$ on the obvious basis $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$ of $V^{\otimes d}$. Let $(a_1, a_2, \ldots, a_j)$ be one of the orbits of σ. If we are to map $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}$ to a multiple of itself, we must have $i_{a_1} = i_{a_2} = \cdots = i_{a_d}$.

So the nonzero contributors to $F(\delta_\sigma)$ are indexed by functions $\{\text{orbits of } \sigma\} \to \{1, \ldots, n\}$. Given an orbit Ω of σ, if it is mapped to k, we see that we get a contribution of $t_k^{|\Omega|}$. So

$$F(\delta_\sigma) = \frac{1}{d!} \sum_{\phi : \mathrm{orb}(\sigma) \to \{1, \ldots, n\}} \prod_{\Omega \in \mathrm{orb}(\sigma)} t_{\phi(\Omega)}^{|\Omega|}.$$

We can factor the sum as

$$\prod_{\Omega \in \mathrm{orb}(\sigma)} \left(t_1^{|\Omega|} + t_2^{|\Omega|} + \cdots + t_n^{|\Omega|}\right)$$

which is

$$\prod_{k=1}^{s} (t_1^{\mu_k} + t_2^{\mu_k} + \cdots + t_n^{\mu_k}) = p_\mu(t).$$

In summary,

$$F(\delta_\sigma) = \frac{1}{d!} p_\mu.$$

3.1. **Orthogonality of the power symmetric functions.** Let $M(\mu)$ be the size of the conjugacy class of σ, with $\mu = c(\sigma)$. Let ϵ_μ be the function which is 1 on elements with cycle structure μ, and 0 on all other elements. By linearity

$$F(\epsilon_\mu) = \frac{M(\mu)}{d!} p_\mu.$$

Let $(\ ,\ )$ be the inner product $\frac{1}{d!}\sum_{\sigma \in S_d} f(\sigma)\overline{g}(\sigma)$ on class functions of S_d. Character functions of S_d-irreps are orthonormal for $(\ ,\ )$, and F sends characters of irreps to Schurs, so $(f, g) = \langle F(f), F(g)\rangle$. In particular, for $\lambda \neq \mu$, we have

$$\langle p_\lambda, p_\mu \rangle = \text{constant} \cdot (\epsilon_\lambda, \epsilon_\mu).$$

But the right hand side is clearly zero, since ϵ_λ and ϵ_μ have distinct supports in S_d. So we now have a conceptual explanation for why the power symmetrics are orthogonal.

We can also deduce something interesting by pairing ϵ_μ with itself. On the one hand

$$(\epsilon_\mu, \epsilon_\mu) = \frac{M(\mu)^2}{(d!)^2 z_\mu}$$

in the notation of Problem Set 1. On the other hand, it is clear that

$$(\epsilon_\mu, \epsilon_\mu) = \frac{M(\mu)}{d!}.$$

So

$$\frac{d!}{M(\mu)} = \frac{1}{z_\mu}.$$

Notice that $d!/M(\mu)$ is the size of the centralizer, $Z(\sigma)$, of σ. So $z_\mu = 1/|Z(\sigma)|$. (Some of you pointed out that Stanley defines z_μ to be the reciprocal of what I wrote; this convinces me his definition is better.)

4. The Frobenius character formula

Let χ_λ be the character of $Sp(\lambda)$. So we have

$$\chi_\lambda = \sum_\sigma \chi_\lambda(\sigma)\delta_\sigma$$

as functions on S_d. Applying F to both sides, we deduce

$$s_\lambda = \sum_{\sigma \in S_d} \chi_\lambda(\sigma) \frac{p_{c(\sigma)}}{d!} = \sum_{|\mu|=d} \chi_\lambda(\mu) \frac{M(\mu)}{d!} p_\mu.$$

In other words, the character table of S_d is, up to some minor conversion factors, the change of basis matrix from power symmetric's to Schurs.

In fact, we can make it look nicer by switching the roles of p and s. Using the self-orthogonality of p's and s's, we can compute

$$\chi_\lambda(\mu) = \langle s_\lambda, p_\mu \rangle$$
$$p_\mu = \sum_\lambda \chi_\lambda(\mu) s_\lambda$$

On Problem Set 8, you'll derive a combinatorial formula for $\langle s_\lambda, p_\mu \rangle$.