TENSOR INVARIANTS

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Throughout this note, let $\dim V = d$, and fix two positive integers m and n. We will be discussing $(V^{\otimes m} \otimes (V^{\vee})^{\otimes n})^{SL(V)}$. I'll use v's to denote elements of V and u's for elements of $V^{\vee}$.

1. Fixed vectors versus invariant functions

$V^{\otimes m} \otimes (V^{\vee})^{\otimes n}$ is the same thing as $\mathrm{Hom}\left((V^{\vee})^{\otimes m} \otimes V^{\otimes n}, \mathbb{C}\right)$. So, instead of looking at fixed vectors in $V^{\otimes m} \otimes (V^{\vee})^{\otimes n}$, we could think about $SL(V)$ invariant linear functions on $(V^{\vee})^{\otimes m} \otimes V^{\otimes n}$.

For general analysis, this seems to just add a layer of complication. But for giving examples, it makes things nicer. Let's see why.

Remember that $X \otimes Y$ is spanned by simple tensors; meaning tensors of the form $x \otimes y$. To give an element ϕ of $(X \otimes Y)^{\vee}$ is to give a map which, to any $x \in X$ and $y \in Y$, computes a number $\phi(x \otimes y)$. Moreover, this computation must be linear in both x and in y.

So an element of $\mathrm{Hom}\left((V^{\vee})^{\otimes m} \otimes V^{\otimes n}, \mathbb{C}\right)$ means a map ϕ which takes in m elements of $V^{\vee}$ and n elements of V and, in a multilinear fashion, produces a scalar.

For example, $(u, v) \mapsto \langle u, v\rangle$ is an element of $\mathrm{Hom}(V^{\vee} \otimes V, \mathbb{C})$. If $\dim V = 3$, then $(v_1, v_2, v_3) \mapsto v_1 \wedge v_2 \wedge v_3$ is an element of $\mathrm{Hom}(V^{\otimes 3}, \mathbb{C})$ (where we have chosen an identification $\bigwedge^3 V \cong \mathbb{C}$.) Note that the first function is $GL(V)$ invariant, and the second is $SL(V)$ invariant.

For whatever reason, mathematicians don't have simple names for "the $GL(V)$ invariant element of $V \otimes V^{\vee}$", or for "the $SL(V)$ invariant element of $(V^{\vee})^{\otimes 3}$ when $\dim V = 3$." Even if you have names for these, you probably don't have a name for the element of $V^{\otimes 3} \otimes (V^{\vee})^{\otimes 3}$ which corresponds to

$$(u_1, u_2, u_3, v_1, v_2, v_3) \mapsto \det \begin{pmatrix} \langle u_1, v_1\rangle & \langle u_1, v_2\rangle & \langle u_1, v_3\rangle \\ \langle u_2, v_1\rangle & \langle u_2, v_2\rangle & \langle u_2, v_3\rangle \\ \langle u_3, v_1\rangle & \langle u_3, v_2\rangle & \langle u_3, v_3\rangle \end{pmatrix}.$$

So, when we start talking about particular invariant elements in tensor products, it's very natural to switch to talking about multilinear functionals. I hope you can reach the point that the two languages seem interchangeable.

Note that multiplication corresponds to tensor here. For example, let E be the element of $V \otimes V^{\vee}$ which corresponds to the map $(u, v) \mapsto \langle u, v\rangle$. (If e_i and e^i are dual bases for V and $V^{\vee}$, then $E = \sum_i e_i \otimes e^i$.) Then $E \otimes E$, in $V \otimes V^{\vee} \otimes V \otimes V^{\vee}$, corresponds to the map $(u_1, v_1, u_2, v_2) \mapsto \langle u_1, v_1\rangle\langle u_2, v_2\rangle$.

2. Changing from duals to adjugates

Let $W = V^{\vee} \otimes \det$. This is the irreducible $GL(V)$ representation with character $s_{1^{d-1}}$. As an $SL(V)$ representation, it is isomorphic to $V^{\vee}$, since $SL(V)$ acts trivially on det. We can also describe W as $\bigwedge^{d-1}(V)$. The reason I bring up W is that, unlike $V^\vee$, it is a polynomial representation, so it fits more easily into our symmetric function techniques.

So $(V^{\otimes m} \otimes (V^\vee)^{\otimes n})^{SL(V)}$ is isomorphic to $(V^{\otimes m} \otimes W^{\otimes n})^{SL(V)}$. Now, $V^{\otimes m} \otimes W^{\otimes n}$ is a polynomial $GL(V)$ representation, on which $t \cdot \mathrm{Id}$ acts by $t^{m+(d-1)n}$. Since any element of $GL(V)$ is a product of an element of $SL(V)$, and an element of the form $t \cdot \mathrm{Id}$, we see that $GL(V)$ acts on $(V^{\otimes m} \otimes W^{\otimes n})^{SL(V)}$ by $\det^{(m+(d-1)n)/d}$.

So

$$\left(V^{\otimes m} \otimes W^{\otimes n}\right)^{SL(V)} \cong \mathrm{Hom}_{GL(V)}(\det^{(m+(d-1)/n)/d}, V^{\otimes m} \otimes W^{\otimes n}).$$

In particular, it is nonzero only if d divides $m + (d-1)n$ or, equivalently, if $m \equiv n \mod d$.

For notational convenience, we set $r = \frac{m+(d-1)n}{d}$.

3. Computation with symmetric polynomials

We see that the dimension of the space $(V^{\otimes m} \otimes W^{\otimes n})^{SL(V)}$ is the same as the multiplicity of $(\det)^r$ in $V^{\otimes m} \otimes W^{\otimes n}$. In other words, this is the coefficient of s_{r^d} in $s_1^m s_{1^{d-1}}^n = e_1^m e_{d-1}^n$. You could compute this using Pieri's rule.

I should say a little about working in Λ versus working in Λ_d. The ring Λ_d encodes the structure of tensor products of polynomial GL_d representations. So the multiplicity of $V_\nu(d)$ in $V_\lambda(d) \otimes V_\mu(d)$ is the same as the coefficient when $s_\lambda(x_1, x_2, \ldots, x_d) s_\mu(x_1, \ldots, x_d)$ is expanded in the Schur basis.

The map $\Lambda \to \Lambda_d$ is a map of rings. Its kernel is $\mathrm{Span}_{\mathbb{Z}}(s_\nu)_{\ell(\nu)>d}$. So it is fine to multiply $s_\lambda s_\mu$ in Λ and read off the coefficient of s_ν there. You'll get the same result, since the map $\Lambda \to \Lambda_d$ sends $s_\nu \to s_\nu$ if $\ell(\nu) \leq d$, and sends s_ν to 0 if $\ell(\nu) > d$.

3.1. So, how does the inner product fit in?

In this course, I only defined the inner product on Λ, not Λ_d. Perhaps it would have been clearer to define it on Λ_d as well: Make the definition be that $(s_\lambda)_{\ell(\lambda) \leq d}$ is an orthonormal basis for Λ_d. This will ensure that $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_{GL}(V, W)$.

Then the map $\Lambda \to \Lambda_d$ is norm preserving *on the subspace spanned by* $(s_\lambda)_{\ell(\lambda) \leq d}$. So we may transfer inner product computations from Λ to Λ_d if they do not involve partitions with more than d parts. More specifically, it is enough for one of the two sides of the inner product to involve no such terms.

It is convenient to note that $\mathrm{Span}(s_\lambda)_{\ell(\lambda) \leq n} = \mathrm{Span}(h_\lambda)_{\ell(\lambda) \leq n}$.

Let's demonstrate a WRONG computation. Suppose we wanted to find $\dim(V^{\otimes n} \otimes (V^\vee)^{\otimes n})^{GL(V)}$. This is the same as $\dim \mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n})$. It would NOT be right to say that this is $\langle s_1^n, s_1^n \rangle$, unless we knew that d was large enough that s_1^n contained not partitions with more than d parts. (Specifically, we would need $d \geq n$. Conveniently for you, in Problem 3.(a), Problem Set 6, I specified $\dim V \geq n$.)

On the other hand, computing this quantity as $\langle s_1^n s_{1^{d-1}}^n, s_{n^d} \rangle$, as we described above, is fine. The quantity s_{n^d} does not contain any partitions with more than d parts.