NOTES FOR NOVEMBER 12, 2012: CATCH-UP DAY

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1. Questions

This class was a catch-up day motivated by student-asked questions. The questions are presented in Section 1. In the remaining sections, we answer each question (not necessarily in order).

1. Where is $V_\lambda^\vee$ in $\mathbb{C}[z_{ij}]$?
2. We have $C_{\lambda\mu}^\nu = \mathrm{Hom}_{\mathrm{GL}}(V_\nu, V_\lambda \otimes V_\mu)$, and $V_\lambda \otimes V_\mu = \bigoplus C_{\lambda\mu}^\nu \otimes V_\nu$. How do those tie together again?
3. Relation to $\mathrm{Hom}(V_\lambda \otimes V_\mu, V_\nu)$?
4. How do we pass between S_d-representations and GL_n-representations?
5. Gelfand-Tsetlin Bases?
6. Why can't we pick any weight vector in V and just map it to a vector of the same weight in W?

2. Isotypic Components (Questions 2, 3, 6)

Let K be a group and W be a K-representation which we know to be a direct sum of simples. Let V be a K-simple. We know that

$$\dim \mathrm{Hom}_K(V, W) = \text{number of times } V \text{ occurs in } W.$$

Let $C = \mathrm{Hom}_K(V, W)$. We have a natural map

$$C \otimes V \to W.$$

This is injective. Why? We can write

$$W = \underbrace{V \oplus \cdots \oplus V}_{n \text{ copies}} \oplus X,$$

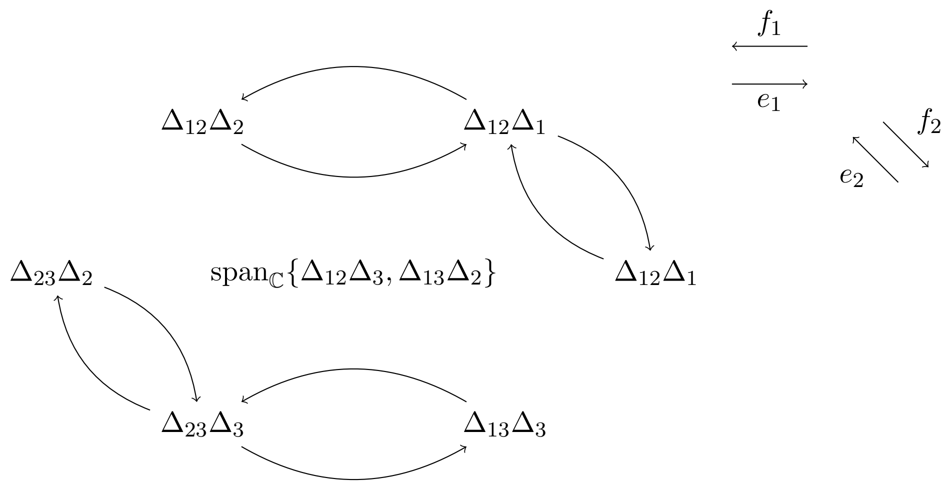
with no V's in X. (Note that this requires a choice of decomposing the V-isotypic part into a direct sum of n copies of V!) Therefore we have

$$W \cong \bigoplus_{V \text{ a } K\text{-simple}} \mathrm{Hom}_K(V, W) \oplus V.$$

This is as canonical of a decomposition as possible. This answers Question 2.

Now, $\mathrm{Hom}_K(W, V) \cong \mathrm{Hom}_K(V, W)^\vee$. Another way to frame this is that $\langle A, B \rangle = Tr(AB)$ is a perfect pairing between $\mathrm{Hom}_K(W, V)$ and $\mathrm{Hom}_K(V, W)$. This answers Question 3.

Suppose we have $V = W = V_{21}(3)$. By Schur's lemma, $\mathrm{Hom}_{GL_3}(V, W)$ is one-dimensional. The $(1, 1, 1)$ weight space is 2 dimensional. So we should NOT be able to just take any vector in the $(1, 1, 1)$ weight space and send it to any other. For example, if we tried to send $\Delta_{12}\Delta_3$ to $\Delta_{13}\Delta_2$, this would not work because $\Delta_{12}\Delta_3$ is negated by $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, whereas $\Delta_{13}\Delta_2$ is not. This answers Question 6.



3. Identifying $V_\lambda^\vee$ in $\mathbb{C}[z_{ij}]$ (Question 1)

We have

$$\mathbb{C}[z_{ij}] \cong \bigoplus_\lambda V_\lambda^\vee \otimes V_\lambda$$

as a $\mathrm{GL} \times \mathrm{GL}$-representation. As λ ranges through partitions with $\ell(\lambda) \leq n$, the representations $V_\lambda(n)$ range through the polynomial irreducible representations of GL_n. Recall that $\rho_{V^\vee}(g) = \rho_V(g)^{-\mathrm{t}}$. Therefore, if V and $V^\vee$ are almost never simultaneously polynomial representations.

Recall that every rational representation of GL_n is of the form $(\det)^{-N} \otimes V_\lambda(n)$ since

$$V_{\lambda+(1,\ldots,1)}(n) \cong V_\lambda(n) \otimes (\det).$$

If we index rational GL_n-representations by sequences $\mu_1 \geq \cdots \geq \mu_n$ of integers, then

$$V^\vee_{\mu_1\cdots\mu_n}(n) = V_{(-\mu_1)\cdots(-\mu_n)}(n).$$

Why is this true? On the level of characters, we have

$$\chi_{V^\vee}(t_1, \ldots, t_n) = \chi_V(t_1^{-1}, \ldots, t_n^{-1}).$$

This explains the negation. Because we label our representations by the highest, then the order will reverse because of this negation. In terms of the indexing, this is where the dual appears.

An example of this is the following. Let $n = 3$, $\mu = (4, 2, 1)$. Then we have

$$V_{421}^\vee(3) \cong V_{(-1)(-2)(-4)}(3) \cong (\det)^{-4} \otimes V_{320}(3).$$

There was a homework problem about working this out.

If we want to see $V_\lambda^\vee$ in $\mathbb{C}[z_{ij}]$, what we will see is the following: Inside $\mathbb{C}[z_{ij}]$ with a right action, you will see $V_{(N-\lambda_1,\ldots,N-\lambda_n)}$ for $N \geq \lambda_1$.

Of course, $V_\lambda^\vee$ does turn up for the left action, because the left action has an inverse built into it.

4. The Relationship Between Representations of S_d and GL_n (Questions 4 and 5)

The big statement of Schur-Weyl duality is the following. We have

$$V^{\otimes d} \cong \bigoplus_{|\lambda|=d} \mathrm{Sp}(\lambda) \otimes V_\lambda(n)$$

for $\dim V = n$. Note that here, $\mathrm{Sp}(\lambda)$ is the Specht module corresponding to the partition λ. We have

$$V_\lambda(n) \cong \mathrm{Hom}_{S_d}(\mathrm{Sp}(\lambda), V^{\otimes d}).$$

We have

$$V_d(n) \cong \mathrm{Hom}_{S_d}(\mathbb{C}, V^{\otimes d}) \cong (V^{\otimes d})^{S_d} = \mathrm{Sym}^d(V).$$

Doing this for more complicated partitions λ, we would recover the Young symmetrizer.

If $\ell(\lambda) \le n$, then

$$\mathrm{Sp}(\lambda) = \mathrm{Hom}_{\mathrm{GL}_n}(V_\lambda(n), V^{\otimes d}).$$

Consider

$$V_{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}} = \mathrm{Hom}_{S_d}(\mathrm{sgn}, V^{\otimes d}).$$

This is the set of $\eta \in V^{\otimes n}$ such that switching the order of the $\otimes$s in $V^{\otimes d}$ acts on η by $\mathrm{sgn} = -1$. So this is the set of anti-symmetric tensors. Thus,

$$V_{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}} = \Lambda^d V.$$

Then

$$\mathrm{Hom}_{\mathrm{GL}_n}(V_{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}}, V^{\otimes d}) = \mathrm{Hom}_{\mathrm{GL}_n}(\Lambda^d V, V^{\otimes d}).$$

This is a one-dimensional space and is explicitly given by sending $v_1 \wedge \cdots \wedge v_d \mapsto c \sum (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$. If we act on $V^{\otimes d}$ by $\tau \in S_d$, then $c \mapsto (-1)^\tau c$.