NOTES FOR NOV 14

STEFAN FROEHLICH

Lie Algebras and Lie Groups

Throughout these notes, G is a smooth Lie group and $\mathfrak{g}$ is $T_{Id}G$. For $g \in \mathfrak{g}$ think about $1 + \epsilon g$ as an element of G near Id of G. Our main focus will be for when $G = GL_n$. In this case $\mathfrak{gl}_n$ is the set of $n \times n$ matrices.

If we have a smooth representation $\rho : G \to GL_N$ and $g \in \mathfrak{g}$, then $\rho(Id + \epsilon g) = Id + \epsilon\sigma(g) + O(\epsilon^2)$ for some linear map $\sigma : \mathfrak{g} \to \mathfrak{gl}_N$.

Proposition: If G is connected, then the map σ determines ρ.

We wll provide a proof for GL_n but will point out how to generalize this to any general Lie group.

Proof: Suppose $\rho_1, \rho_2 : GL_n \to GL_N$ are two representations of GL_n such that $\rho_1(1 + \epsilon g)$ and $\rho_2(1 + \epsilon g)$ are both of the form $1 + \epsilon\, \sigma(g) + O(\epsilon^2)$.

Let U be a small convex set containing 0 in $\mathfrak{g}$ such that $Id + U \subset GL_n$ (More generally, choose a convex subset U containing 0 in $\mathfrak{g}$ and choose a smooth ϕ such that $\phi(0) = e$ and $(D\phi)_0 = Id$).

For $g \in U$, $\rho_1((1 + \frac{g}{n})^n) = \rho_1(1 + \frac{g}{n})^n = (1 + \frac{\sigma(g)}{n} + O(\frac{1}{n^2}))^n$, so

$$\rho_1\left(\lim_{n\to\infty}\left(1 + \frac{g}{n}\right)^n\right) = \lim_{n\to\infty}\left(1 + \frac{\sigma(g)}{n} + O\left(\frac{1}{n^2}\right)\right)^n = \lim_{n\to\infty}\left(1 + \frac{\sigma(g)}{n}\right)^n$$

where we were able to move the limit outside of ρ_1 since ρ_1 is continuous by assumption.
(In the general case, $\rho_1\left(\lim_{n\to\infty}\phi(\frac{g}{n})^n\right) = \lim_{n\to\infty}\left(1 + \frac{\sigma(g)}{n}\right)^n$).

This leads us to the definition of the exponential map. For any matrix $A \in \mathfrak{gl}_n$, set $\exp(A) := \lim_{n\to\infty}(1 + \frac{A}{n})^n$.

Expanding the expression in the definition, we find that:

$$\exp(A) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{A^k}{k!}\frac{n(n-1)\cdots(n-k+1)}{n^k} = \sum_{k=0}^{\infty}\lim_{n\to\infty}(\cdots) = \sum_{k=0}^{\infty}\frac{A^k}{k!}$$

where we could switch the order of the sum and limit by general analysis arguments. The entries A^k grow at most exponentially, so this infinite sum will converge for any matrix A.

Expressing the earlier result using the exponential map, we find that for $g \in U$, $\rho_1(\exp(g)) = \exp(\sigma(g))$ and the argument also gives $\rho_2(\exp(g)) = \exp(\sigma(g))$.

Shrinking U if necessary, we can arrange it so that $\exp : U \to G$ is an open map. Let $X = \{\gamma \in G : \rho_1(\gamma) = \rho_2(\gamma)\}$. X is closed since ρ_1 and ρ_2 are continuous. For any $x \in X$, $\rho_1(x \exp(U)) = \rho_1(x)\rho_1(\exp(U)) = \rho_2(x)\rho_2(\exp(U)) = \rho_2(x \exp(U))$ so $x \exp(U) \subset X$. Each point of X has an open neighborhood contained in X, so X is open.

G is connected and $X \subset G$ is both open and closed, so we must have that $X = G$ or $X = \emptyset$. $Id \in X$ so $X = G$ and $\rho_1 = \rho_2$.

Representations of Lie algebras

A Lie algebra is a vector spce $\mathfrak{g}$ along with a bilinear map $[\,,\,] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ such that $[g, h] = -[h, g]$ and $[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$ (this is called the Jacobi Identity).

Example: $\mathfrak{g} = \mathfrak{gl}_n$ where $[\,,\,]$ is the commutater.

By the proposition, any smooth representation is determined by how σ acts on its Lie algebra. The next natural question then is what can we say about σ that arise as the linear term of a representation. Again our work will be for $\mathfrak{g}l_n$ though it is true for all $\mathfrak{g}$.

Consider the product $\exp(g\epsilon)\exp(h\epsilon)\exp(-g\epsilon)\exp(-h\epsilon)$

Expanding each exponential as its defining series, we get that this product equals:

$$\begin{aligned}
&= (1 + g\epsilon + \frac{1}{2}g^2\epsilon^2 + \cdots)\cdots(1 - h\epsilon + \frac{1}{2}h^2\epsilon^2 + \cdots) \\
&= 1 + \left[\left(\frac{g^2}{2} - g^2 + \frac{g^2}{2}\right) + (gh - gh - hg + gh) + \left(\frac{h^2}{2} - h^2 + \frac{h^2}{2}\right)\right]\epsilon^2 + O(\epsilon^3) \\
&= 1 + [g,h]\epsilon^2 + O(\epsilon^3))
\end{aligned}$$

where $[g,h] = gh - hg$ is the commutator of g and h.

Using one of the results shown while proving the proposition, we find that

$$\begin{aligned}
\rho(\exp(g\,\epsilon)\exp(h\,\epsilon)\exp(-g\,\epsilon)\exp(-h\,\epsilon)) &= \exp(\epsilon\,\sigma(g))\exp(\epsilon\,\sigma(h))\exp(-\epsilon\,\sigma(g))\exp(-\epsilon\,\sigma(h)) \\
&= 1 + [\sigma(g),\sigma(h)]\epsilon^2 + O(\epsilon^3).
\end{aligned}$$

We also have that $\rho(\exp(g\,\epsilon)\cdots\exp(-h\,\epsilon)) = \rho(1 + [g,h]\epsilon^2 + O(\epsilon^3)) = 1 + \sigma([g,h])\epsilon^2 + O(\epsilon^3)$. So $\sigma([g,h]) = [\sigma(g),\sigma(h)]$.

Alternatively, we could have gotten this result without having to use exp by looking at

$$\phi(\delta g)\phi(\epsilon h)\phi(-\delta g)\phi(-\epsilon h) = 1 + (\ \ )\delta^2 + [g,h]\delta\epsilon + (\ \ )\epsilon^2 + O(\delta + \epsilon)^3$$

for an arbitrary ϕ and then perform the same argument using this expansion.

Our result on σ leads us to the following definition:
Definition: A representation of $\mathfrak{g}$ is a linear map $\sigma : \mathfrak{g} \to \mathfrak{g}l_N$ such that $\sigma([g,h]) = [\sigma(g),\sigma(h)]$.