NOTES FOR NOVEMBER 16

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Let's recall our setup from last time. We have the Lie group GL_n, with Lie algebra

$$\mathfrak{gl}_n = \{n \times n \text{ matrices}\}.$$

There is a map $[\,,\,] : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathfrak{gl}_n$ defined by $[A, B] = AB - BA$. For each representation $\rho : GL_n \to GL_N$, we have a corresponding linear map $\sigma : \mathfrak{gl}_n \to \mathfrak{gl}_N$, with $\sigma([A, B]) = [\sigma(A), \sigma(B)]$.

Today, let's look at how $\mathfrak{gl}_2$ fits in to the larger $\mathfrak{gl}_n$. Let e_{ij} be the elementary matrix with a 1 in the i^{th} row and j^{th} column, and 0's everywhere else. We have

$$\begin{aligned}[e_{ij}, e_{k\ell}] &= e_{i\ell}\delta_{jk} - e_{jk}\delta_{i\ell} \\ &= e_{i\ell} \text{ (if } j = k) - e_{jk} \text{ (if } i = \ell)\end{aligned}$$

We define $h_k = e_{kk}$, $e_k = e_{k(k+1)}$, $f_k = e_{(k+1)k}$.

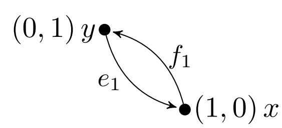
Note: the set $\{e_k, f_k, h_k\}$ generates $\mathfrak{gl}_n$. For instance, we have

$$\begin{aligned}[e_1, [e_2, e_3]] &= [e_{12}, [e_{23}, e_{34}]] \\ &= [e_{12}, e_{24}] \\ &= e_{14}\end{aligned}$$

Let's look at $\mathfrak{gl}_2$ acting on $\mathbb{C}^2 = \mathbb{C} \cdot \{x, y\}$.

$$h_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} h_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} f_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We can summarize this with a picture:



For $V_2(2) = \mathbb{C} \cdot \{x^2, xy, y^2\}$, we have

$$\left(1 + \epsilon \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) \cdot \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix} = \begin{pmatrix} x^2 + 2a\epsilon x^2 + \cdots \\ xy + \epsilon(a+b)xy + \cdots \\ y^2 + 2b\epsilon y^2 + \cdots \end{pmatrix}$$

So

$$\sigma \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 2a & & \\ & a+b & \\ & & 2b \end{pmatrix}$$

For example

$$\left(1 + \epsilon \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \cdot \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix} = \begin{pmatrix} x^2 + \cdots \\ xy + \epsilon x^2 + \cdots \\ y^2 + 2\epsilon xy + \cdots \end{pmatrix}$$

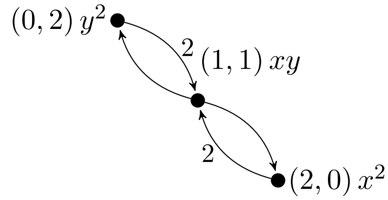
So we have

$$e_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

and similarly

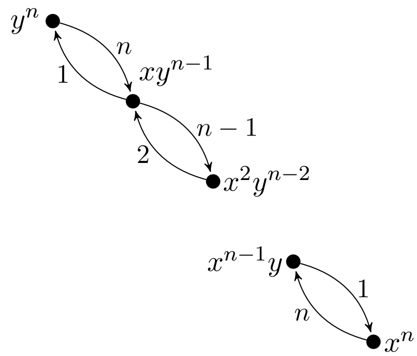
$$f_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The corresponding picture is

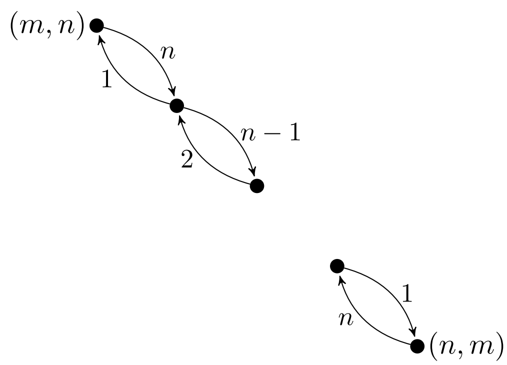


The downward arrows correspond to the action of e_1, the upward arrows to f_1.

For $\mathrm{Sym}^n\mathbb{C}^2$, we get a similar picture



For $V_{nm}(2) = (\det)^{\otimes m}V_{n-m}(2)$, we get the same picture, only shifted.



1. Weight spaces

In the above, we have used weight spaces to organize our pictures. Now we make some general statements about how weight spaces will behave. First, we relate behavior of weights for the Lie group action to weights for the Lie algebra representation.

If

$$\rho\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \cdot v = t_1^{k_1} \cdots t_n^{k_n} v$$

then we have

$$\sigma\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \cdot v = (k_1a_1 + \cdots + k_na_n)v$$

Proposition: If u is in the $(k_1, k_2, \ldots, k_n)$ weight space, then $\sigma(E_{ij})u$ is in the

$$(k_1, k_2, \ldots, k_i + 1, \ldots, k_j - 1, \ldots, k_n)$$

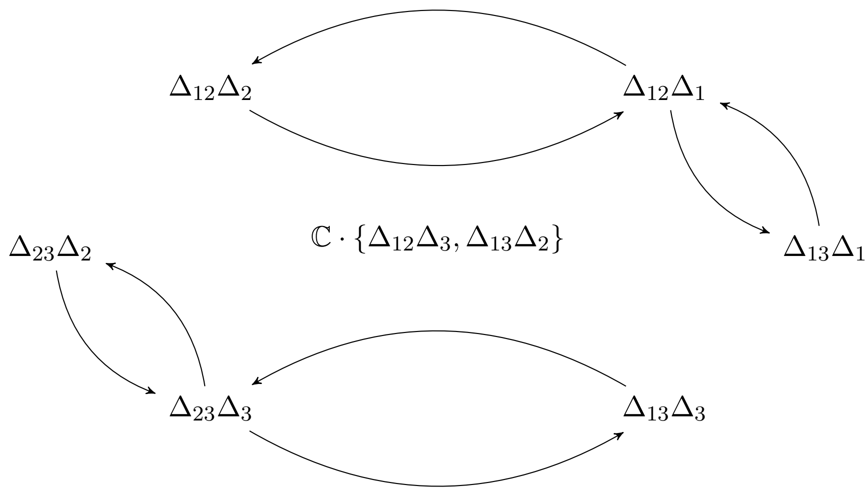
weight space.

Proof:

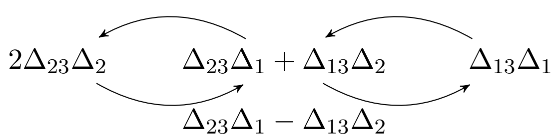
$$\sigma\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}(E_{ij})u = \left(\sigma(E_{ij})\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix} + \sigma\left[\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}, E_{ij}\right]\right)u$$
$$= (k_1a_1 + \cdots + k_na_n)\sigma(E_{ij})u + \sigma((a_i - a_j)E_{ij})u$$
$$= (k_1a_1 + \cdots + k_na_n)\sigma(E_{ij})u + (a_i - a_j)\sigma(E_{ij})u$$

So e_i increases the i^{th} component, and decreases the $(i+1)^{st}$, while f_i decreases the i^{th} component and increases the $(i+1)^{st}$.

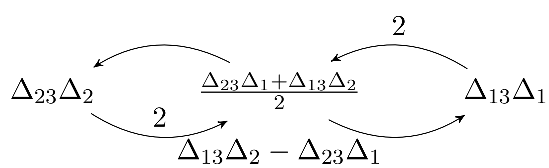
Now let's look at $V_{21}(3)$. The action of f_1 corresponds to arrows pointing left; e_2 to arrows pointing right, e_2 to arrows pointing up, and f_2 to arrows pointing down.



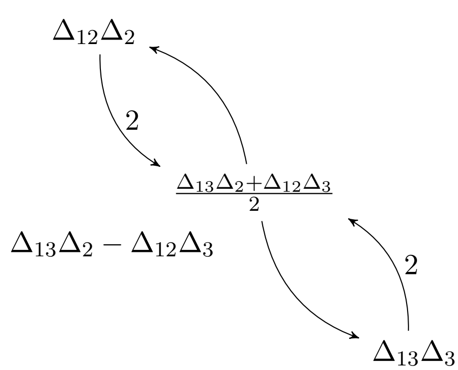
We get a chain



We can normalize this "nicely" by rescaling our basis vectors



Similarly we get a chain



Notice that $\frac{\Delta_{23}\Delta_1+\Delta_{13}\Delta_2}{2}$, $\Delta_{13}\Delta_2 - \Delta_{23}\Delta_1$, $\frac{\Delta_{13}\Delta_2+\Delta_{12}\Delta_3}{2}$ and $\Delta_{13}\Delta_2 - \Delta_{12}\Delta_3$ are genuinely four different vectors inside $V_{21}(3)$. Even if we decided to take two of them as our preferred basis, making half of the maps in and out of the $(1,1,1)$ weight space nice, the other maps would still have nontrivial matrices.