NOTES FOR NOVEMBER 19, 2012: INTERACTION BETWEEN $\mathfrak{gl}_2$ STRINGS

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1. From last class ($\mathfrak{gl}_2$ representations)

The Lie algebra $\mathfrak{gl}_n$ consists of the $n \times n$ matrices with bracket $[\,,] : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathfrak{gl}_n$ given by the matrix commutator $[A, B] = AB - BA$. It has a basis of elementary matrices E_{ij} with a single 1 in ith row and jth column, and 0's everywhere else.

Define

$$h_k = E_{k\,k} \qquad e_k = E_{k\,(k+1)} \qquad f_k = E_{(k+1)\,k}.$$

In particular,

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are elements of $\mathfrak{gl}_2$.

For any smooth representation $\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$, the differential of ρ is a Lie algebra representation $\sigma : \mathfrak{gl}_n \to \mathfrak{gl}(V) = \mathrm{End}(V)$. If $v \in V$ is in the $(p_1, \ldots, p_n)$ weight space, that is, if

$$\rho \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \cdot v = (t_1^{p_1} \cdots t_n^{p_n}) v,$$

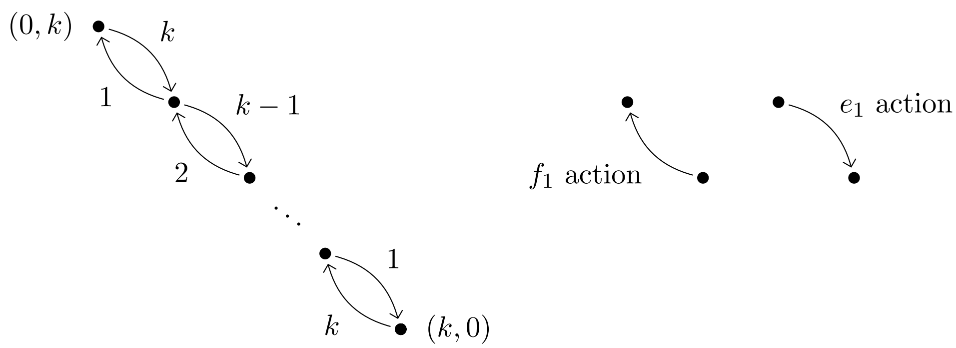
then $\sigma(E_{ij})v$ is in the $(p_1, p_2, \ldots, p_i + 1, \ldots, p_j - 1, \ldots, p_n)$ weight space.

Consider the GL_2-representation $V_{(k)}(2)$ and the corresponding $\mathfrak{gl}_2$-representation. It has a weight basis with weights $(k, 0), (k-1, 1), \ldots, (0, k)$. In that (ordered) basis, the $\mathfrak{gl}_2$-representation is given by

$$\sigma : \mathfrak{gl}_2 \to \mathrm{End}(V_{(k)}(2)) \quad h_1 \mapsto \begin{pmatrix} k & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \qquad h_2 \mapsto \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & k \end{pmatrix}$$

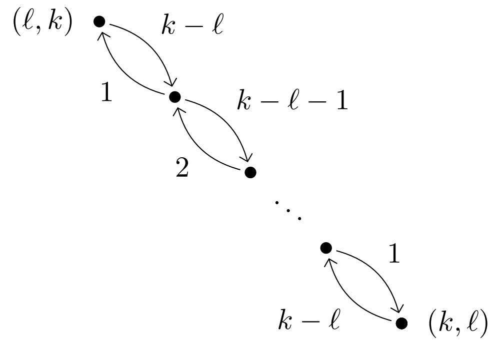
$$e_1 \mapsto \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & & & \ddots & k \\ & & & & 0 \end{pmatrix} \qquad f_1 \mapsto \begin{pmatrix} 0 & & & & \\ k & \ddots & & & \\ & \ddots & 0 & & \\ & & 2 & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

The actions of e_1 and f_1 on $V_{(k)}(2)$ are captured in the following picture



where the dots ($\bullet$) are weight spaces and the arrows for e_1 and f_1 are labeled with their factor.

For the representation $V_{(k,\ell)}(2) \cong (\det)^{\otimes \ell} \otimes V_{(k-\ell)}$, the picture for $V_{(k-\ell)}$ is shifted to the weights $(k,\ell),(k-1,\ell+1),\ldots,(\ell,k)$.



Remark: $\mathfrak{sl}_2$ controls the world!

2. Interaction of $\mathfrak{gl}_2$ strings

How do (e_j, f_j) act on (e_k, f_k) strings of a $\mathfrak{gl}_n$-representation?

Proposition 1. *For any $\mathfrak{gl}_n$-representation:*

- *If $|j-k| \geq 2$, then (e_j, f_j) preserves the length of the (e_k, f_k) string.*
- *If $|j-k| = 1$, then e_j and f_j can only map between strings whose lengths differ by ± 1.*

Proof. Let (σ, W) be a $\mathfrak{gl}_n$-representation. Define

$$\phi_i : \mathfrak{gl}_2 \to \mathfrak{gl}_n$$

by inserting the 2×2 matrix in the rows i and $i+1$ and the columns i and $i+1$, with everything else zero.

$$\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & * & * & & \\ & & * & * & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Suppose that $|j-k| \geq 2$. Then $[\phi_j(\mathfrak{gl}_2), \phi_k(\mathfrak{gl}_2)] = 0$. Any two matrices $u \in \phi_j(\mathfrak{gl}_2)$ and $v \in \phi_k(\mathfrak{gl}_2)$ must commute, so $\sigma(u) : W \to W$ is a map of $\phi_k(\mathfrak{gl}_2)$-representations. So the $\phi_k(\mathfrak{gl}_2)$-isotypic components of W are invariant under $\sigma(u)$, and $\sigma(u)$ preserves the length of (e_k, f_k) strings.

The proof of the second statement uses the Serre relation, which will be introduced later. $\square$

Example 2. Consider $V_{(2,1)}(3)$. It has weight spaces

$$\begin{array}{ccccc} & (1,2,0) & & (2,1,0) & \\ \\ (0,2,1) & & (1,1,1) & & (2,0,1) \\ \\ & (0,1,2) & & (1,0,2) & \end{array}$$

and all of the weight spaces are one-dimensional except for $(1,1,1)$, which is two-dimensional.

The $(1,1,1)$ weight space contains the semistandard basis elements $\{\Delta_{12}\Delta_3, \Delta_{13}\Delta_2\}$. If we instead split the $(1,1,1)$ weight space into the basis $\{\frac{1}{2}(\Delta_{23}\Delta_1 + \Delta_{13}\Delta_2), \frac{1}{2}(\Delta_{23}\Delta_1 - \Delta_{13}\Delta_2\})$, we have the picture

$$\begin{tikzcd}
& \Delta_{12}\Delta_2 \arrow[rr, bend left] & & \Delta_{12}\Delta_1 \arrow[ll, bend left] \arrow[dr, dashed, bend left] & \\
\Delta_{23}\Delta_2 \arrow[rr, bend left, "2"] \arrow[ddr, dashed, bend left] & & \frac{1}{2}(\Delta_{23}\Delta_1 + \Delta_{13}\Delta_2) \arrow[ll, bend left, "1"] \arrow[rr, bend left, "1"] & & \Delta_{13}\Delta_1 \arrow[ll, bend left, "2"] \arrow[ul, dashed, bend left] \\
& & \frac{1}{2}(\Delta_{13}\Delta_2 - \Delta_{23}\Delta_1) & & \\
& \Delta_{23}\Delta_3 \arrow[rr, bend left] \arrow[uul, dashed, bend left] & & \Delta_{13}\Delta_3 \arrow[ll, bend left] &
\end{tikzcd}$$

and with the basis $\{\frac{1}{2}(\Delta_{13}\Delta_2 + \Delta_{12}\Delta_3), \frac{1}{2}(\Delta_{13}\Delta_2 - \Delta_{12}\Delta_3)\}$, we have the picture

$$\begin{tikzcd}
& \Delta_{12}\Delta_2 \arrow[rr, bend left] \arrow[dr, dashed, bend left, "2"] & & \Delta_{12}\Delta_1 \arrow[ll, bend left] \arrow[dr, dashed, bend left] & \\
\Delta_{23}\Delta_2 \arrow[ddr, dashed, bend left] & & \frac{1}{2}(\Delta_{13}\Delta_2 + \Delta_{12}\Delta_3) \arrow[ul, dashed, bend left, "1"] \arrow[ddr, dashed, bend left, "1"] & & \Delta_{13}\Delta_1 \arrow[ul, dashed, bend left] \\
& & \frac{1}{2}(\Delta_{13}\Delta_2 - \Delta_{12}\Delta_3) & & \\
& \Delta_{23}\Delta_3 \arrow[rr, bend left] \arrow[uul, dashed, bend left] & & \Delta_{13}\Delta_3 \arrow[ll, bend left] \arrow[uul, dashed, bend left, "2"] &
\end{tikzcd}$$

where the horizontal solid arrows are (e_1, f_1) strings and the diagonal dashed arrows are (e_2, f_2) strings.

The (e_1, f_1) and (e_2, f_2) strings come from the restriction of the GL_3-representation $V_{(2,1)}(3)$ to the subgroups

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

both isomorphic to GL_2.

For an example of the computations involved, consider the f_2 string

$$\Delta_{12}\Delta_2 \xrightarrow{f_2} \frac{1}{2}(\Delta_{23}\Delta_1 + \Delta_{13}\Delta_2) \xrightarrow{f_2} \Delta_{13}\Delta_3.$$

Act on $\Delta_{12}\Delta_2$ with the GL_3 action by $(1 + \epsilon f_2)$

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \epsilon & 1 \end{pmatrix} \cdot \left(\begin{vmatrix} z_{11} & z_{12} \\ z_{11} & z_{12} \end{vmatrix} z_{12} \right) &= \begin{vmatrix} z_{11} & z_{12} + \epsilon z_{13} \\ z_{21} & z_{22} + \epsilon z_{23} \end{vmatrix} (z_{12} + \epsilon z_{13}) \\
&= \Delta_{12}\Delta_2 + \epsilon(\Delta_{12}\Delta_3 + \Delta_{13}\Delta_2)
\end{aligned}$$

so that we get $\Delta_{12}\Delta_3 + \Delta_{13}\Delta_2$. The action of the Lie algebra also is a derivation (i.e. satisfies the Liebniz rule), so we can compute the same thing with

$$f_2(\Delta_{12}\Delta_2) = \Delta_{12}f_2(\Delta_2) + f_2(\Delta_{12})\Delta_2 = \Delta_{12}\Delta_3 + \Delta_{13}\Delta_2.$$

Note that it has the correct factor of 2 times the basis element.

The second segment of the string is

$$\begin{aligned} f_2\big(\tfrac{1}{2}(\Delta_{12}\Delta_3 + \Delta_{13}\Delta_2)\big) &= \tfrac{1}{2}\big(f_2(\Delta_{12})\Delta_3 + \Delta_{12}f_2(\Delta_3) + f_2(\Delta_{13})\Delta_2 + \Delta_{13}f_2(\Delta_2)\big) \\ &= \tfrac{1}{2}\big(\Delta_{13}\Delta_3 + \Delta_{13}\Delta_3\big) \\ &= \Delta_{13}\Delta_3 \end{aligned}$$

as claimed.

From the pictures, one can also verify that the actions of (e_1, f_1) change the string length of (e_2, f_2) by ± 1, and vice versa.

3. Serre relation

Proposition 3 (Serre relation)**.** *For any $\mathfrak{gl}_n$-representation $\sigma : \mathfrak{gl}_n \to \operatorname{End}(W)$, we have*

$$\sigma(f_k)^2\sigma(f_{k+1}) - 2\sigma(f_k)\sigma(f_{k+1})\sigma(f_k) + \sigma(f_{k+1})\sigma(f_k)^2 = 0$$

et cetera. Similar relations hold with f_k and f_{k+1} swapped, and with e's turned into f's.

For notation simplicity, we drop the σ's.

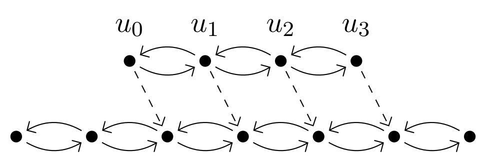
Proof. For the first equations,

$$\begin{aligned} f_k^2 f_{k+1} - 2f_k f_{k+1} f_k + f_{k+} f_k^2 &= [f_k, f_k f_{k+1} - f_{k+1} f_k] \\ &= [f_k, [f_k, f_{k+1}]] \\ &= [E_{(k+1)k}, [E_{(k+1)k}, E_{(k+2)(k+1)}]] \\ &= [E_{(k+1)k}, -E_{(k+2)k}] = 0. \end{aligned}$$

The other equations are similar, and the map of Lie algebras preserves the bracket. $\square$

The Serre relation can be used to show that two (e_k, f_k) strings whose lengths differ by something other than ± 1 cannot be mapped to each other by f_{k+1} (or by e_{k+1}, f_{k-1}, or e_{k-1}).

Proof. Suppose that the target string is at least two longer than the source string. Then at least one end has two extra nodes. Consider this illustration, where left arrows ($\longleftarrow$) are f_k, right arrows ($\longrightarrow$) are e_k, and dashed diagonal arrows ($\searrow$) are f_{k+1}.

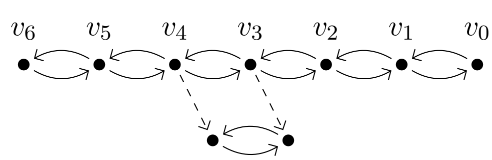


In this picture, in order to get the Serre relation

$$f_k^2 f_{k+1} - 2f_k f_{k+1} f_k + f_{k+1} f_k^2 = 0$$

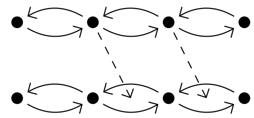
starting from the vector u_0, we must have $f_{k+1} \cdot u_0$. Similarly, $f_{k+1} \cdot u_1 = 0$, and by induction $f_{k+1} u_i = 0$ for all u_i in the top string.

If the target string is at least two shorter than the source string, then we have this picture.



For the Serre relation to hold here, $f_{k+1}(f_k^2(v_0)) = 0$. Again by induction, all f_{k+1} maps are zero.

If the two strings are the same length, the f_{k+1} maps cannot go between the relevant weights.



The above arguments also work for $f_{k+1}, e_{k+1}, f_{k-1}$, and e_{k-1}. Thus, these maps can only map (e_k, f_k) strings to each other if the lengths of the strings differ by ± 1. $\square$