NOTES FOR NOVEMBER 2, 2012: PROOFS

AARON PRIBADI[1]

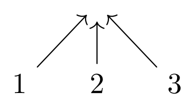
1. Tensor diagrams

Definition 1. A ***tensor diagram*** is a directed graph in a disk such that

- there are $3n$ boundary sources of degree 1;
- all internal vertices are degree 3 sources or degree 3 sinks.

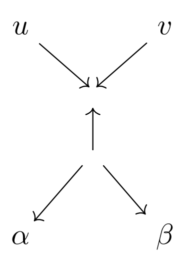
Tensor diagrams correspond to invariants of $(V^{\otimes 3n})^{SL_3} \cong \mathrm{Hom}_{SL_3}(V^{\otimes 3n}, \mathbb{C})$.

Example 2. The tensor diagram

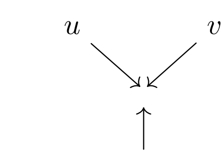


represents the invariant $\Delta_{123} = \det(v_1, v_2, v_3)$.

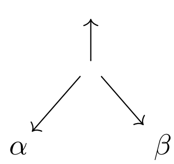
Example 3. Consider the tensor diagram



where we think of tails of arrows as vectors and arrowheads as covectors. We can label the middle arrow with a covector at the head and a vector at the tail by using the isomorphisms $\bigwedge^2 V \cong V^\vee$ and $\bigwedge^2 V^\vee \cong V$ to turn pairs of vectors into a covector and vice-versa:



is the covector $u \wedge v \in \bigwedge^2 V \cong V^\vee$ and



is the vector $\beta \wedge \alpha \in \bigwedge^2 V^\vee \cong V$.

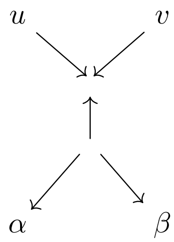
(Why did we take $\beta \wedge \alpha$ and not $\alpha \wedge \beta$? Since the wedge product cares about order, we need to be consistent about how we read off labels—since we read u and v off in clockwise order around the top vertex, we must read β and α off in clockwise order as well.)

The invariant corresponding to this tensor diagram, then, is just the pairing of the resulting covector and vector. However, we do not need to use our isomorphisms $\bigwedge^2 V \cong V^\vee$ and $\bigwedge^2 V^\vee \cong V$

[1]Kevin adds: Because class was a bit rushed in order to get to the elegant ideas at the end, I have added more clarifications and explanations to these notes than usual.

to make sense of a pairing between $u \wedge v \in \bigwedge^2 V$ and $\beta \wedge \alpha \in \bigwedge^2 V^\vee$, as the pairing between V and $V^\vee$ induces a pairing on corresponding exterior powers.

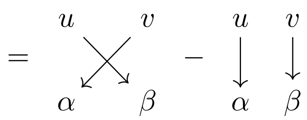
One way or another, we can think about this tensor diagram concretely as a determinant in three slightly different (but necessarily equal) ways, which we can then evaluate and expand as a linear combination of tensor diagrams:



$$= \det \begin{pmatrix} \vdots & \vdots & 1 & 0 & 0 \\ u & v & 0 & 1 & 0 \\ \vdots & \vdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & \beta & \cdots \\ 0 & 0 & \cdots & \alpha & \cdots \end{pmatrix} = \det \left(\begin{pmatrix} \cdots & \beta & \cdots \\ \cdots & \alpha & \cdots \end{pmatrix} \begin{pmatrix} \vdots & \vdots \\ u & v \\ \vdots & \vdots \end{pmatrix} \right)$$

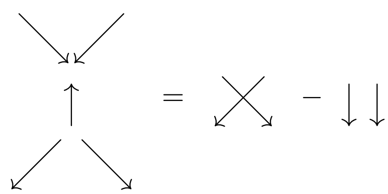
$$= \det \begin{pmatrix} \langle \beta, u \rangle & \langle \beta, v \rangle \\ \langle \alpha, u \rangle & \langle \alpha, v \rangle \end{pmatrix}$$

$$= \langle \beta, u \rangle \langle \alpha, v \rangle - \langle \beta, v \rangle \langle \alpha, u \rangle$$

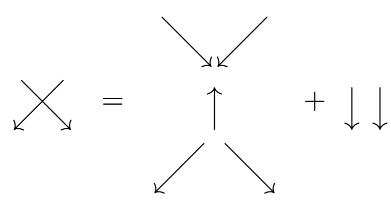


The following are relations of tensor diagrams.

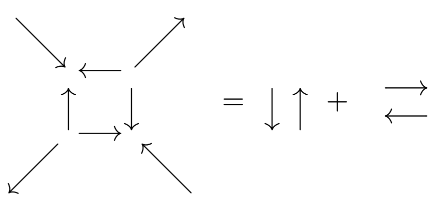
(I)



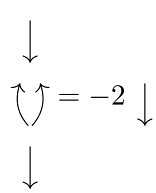
or, equivalently



(II)



(III)



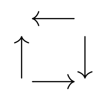
(IV)

 $= 3$

Notice that the relations (II) and (III) follow from (I) and (IV). It is convenient, however, to still include (II) and (III).

Definition 4. A ***web*** is a planar tensor diagram.

Definition 5. A ***non-elliptic*** tensor diagram is a tensor diagram without

 or  .

From the relations (I) – (IV), we can see that the non-elliptic webs span the space of all tensor diagrams; the forbidden shapes can be replaced. Then the (invariants corresponding to) non-elliptic webs span $\mathrm{Hom}_{SL_3}(V^{3n}, \mathbb{C})$.

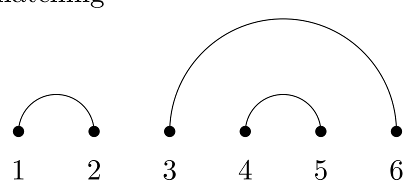
In fact, the non-elliptic webs are in bijection with standard Young tableaux of shape $3 \times n$. Because the number of $3 \times n$ SYT is equal to the dimension of the space, the non-elliptic webs must form a basis. Demonstrating this bijection is our next and final task.

2. Non-crossing tableaux

Recall (from a problem set) that non-crossing matchings on $2n$ elements are in bijection with $2 \times n$ SYT. The non-crossing matchings are also in bijection with $2 \times n$ ***non-crossing tableaux*** (which also have content $1, \dots, 2n$).

From any non-crossing matching, we can construct a $2 \times n$ non-crossing tableau. The columns of the tableau record the matchings.

Example 6. The non-crossing matching



produces the non-crossing tableau

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 5 \\ \hline \end{array}.$$

The column $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ indicates that 1 is matched with 2, the column $\begin{array}{|c|}\hline 3 \\ \hline 6 \\ \hline \end{array}$ indicates that 3 is matched with 6, and the column $\begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline \end{array}$ indicates that 4 is matched with 5.

Any $2 \times n$ tableau that can be constructed in this manner is a non-crossing tableau. This can be generalized to $k \times n$ tableaux.

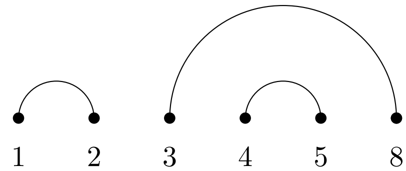
Definition 7. A $k \times n$ ***non-crossing tableau*** (NCT) is a filling of the $k \times n$ shape with content $1, \dots, kn$ exactly once each such that

- the first row and all columns increase, and
- each pair of adjacent rows forms a non-crossing matching.

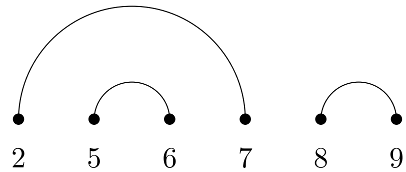
Example 8. The tableau

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 8 & 5 \\ \hline 7 & 9 & 6 \\ \hline \end{array}$$

is a 3×3 NCT because the first two rows correspond with the non-crossing matching



and the last two rows correspond with the non-crossing matching



(the nodes are ordered).

Observe that the bijection between $2 \times n$ SYT and $2 \times n$ NCT gives a bijection between $k \times n$ SYT and $k \times n$ NCT.

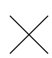
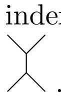
$$\text{SYT} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array} \qquad \longleftrightarrow \qquad \text{NCT} \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 8 & 5 \\ \hline 7 & 9 & 6 \\ \hline \end{array}$$

For more on non-crossing tableaux, see P. Pylyavskyy, "Non-Crossing Tableaux" (arXiv:math/0607211). For even more, see K. Petersen, P. Pylyavskyy, D. Speyer, "A non-crossing standard monomial theory" (arXiv:0806.1776).

3. A bijection between NCT and non-elliptic webs

To show that the non-elliptic webs form a basis for $\mathrm{Hom}_{SL_3}(V^{3n}, \mathbb{C})$, it suffices to give a bijection between $3 \times n$ NCT and non-elliptic webs with $3n$ boundary vertices. We give both directions of a bijection.

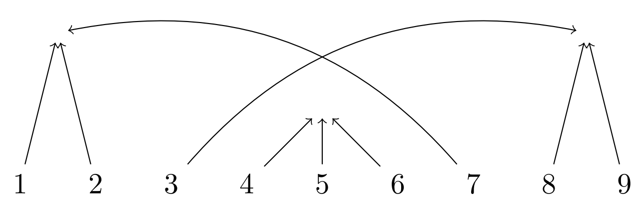
The web map: NCT $\longrightarrow$ non-elliptic web

Start with the "tripod diagram" of a NCT (i.e., the tensor diagram corresponding to the product of determinants indexed by columns of the NCT, drawn without unnecessary crossings), and change crossings  to  .

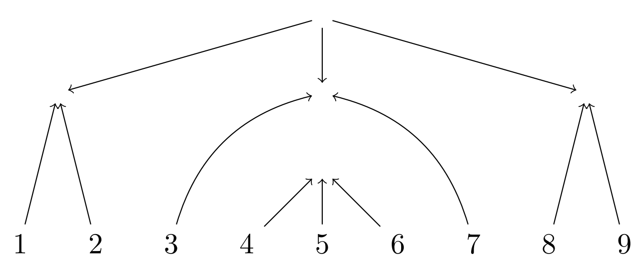
Example 9. Consider the NCT

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 8 & 5 \\ \hline 7 & 9 & 6 \\ \hline \end{array}$$

which has the following tripod diagram.

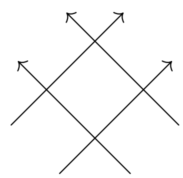


The web map yields the new diagram

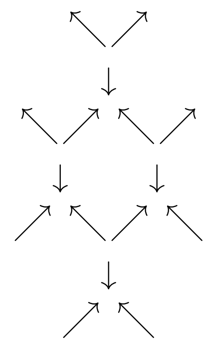


which is a web.

Let's check that the result is non-elliptic; i.e., that all interior faces have at least 6 sides. All of the faces in the tripod diagram have $2n$ sides, with $n \geq 2$, and the web map can't remove sides. If there are already ≥ 6 sides to a face in the tripod diagram, we're good. We can't have any 2-gons in the tripod diagram, as we required them to be drawn without unnecessary crossings. Finally, if we have any faces with 4 sides in the tripod diagram, then we see that the web map must add two more:



becomes



The orientations of the original arrows may differ a bit, but the requirement that all intersections occur between first and third legs of tripods guarantees that our situation looks like this up to rotation.

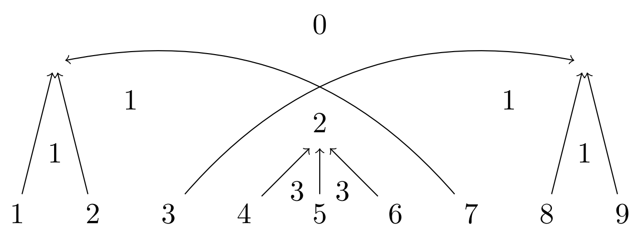
The depth map: non-elliptic web $\longrightarrow$ NCT

How to reverse this? The depth map.

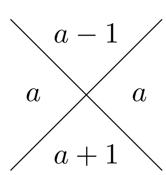
Definition 10. A ***minimal cut path*** between two points in a web diagram is a path between them which crosses the fewest edges.

Definition 11. The ***depth*** of a face is the number of edges crossed by a minimal cut path from outside the web to the face.

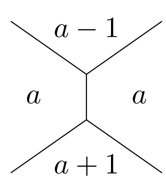
Example 12. The following tripod diagram is annotated with the depths of faces.



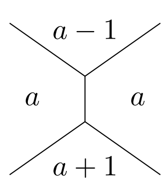
The web map, changing



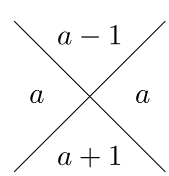
to



does not change the depth of faces. Also, in the tripod picture no internal edge separates faces of the same depth. Hence we can reverse the web map by turning



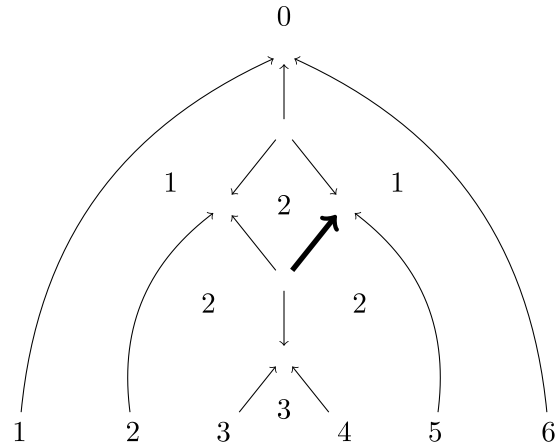
. back into



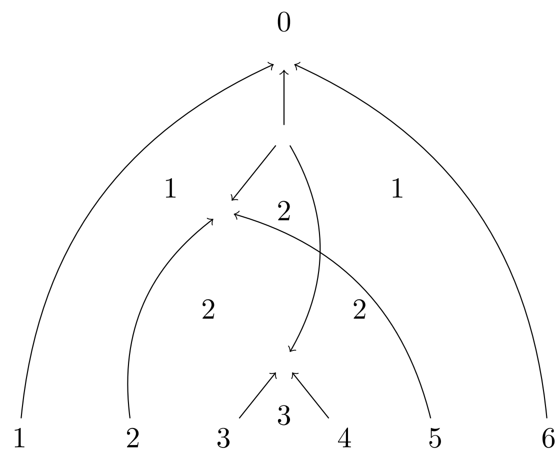
In general, we define the ***depth map*** to be the map from non-elliptic webs to tensor diagrams which replaces all internal edges between faces of the same depth with crossings.

Is this even well-defined? We will see that it is for non-elliptic webs, but it isn't for arbitrary webs:

Example 13 (Cautionary example)**.** Does the depth map make sense starting with an arbitrary web? No.



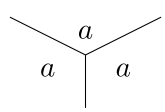
It is not well-defined what to do if edges we want to remove share a vertex! If we try to turn the bold edge between two faces of depth 2 into a crossing, the result is



The new edges separate some faces of the same depth, some with different depths. It's not clear what we should do next.

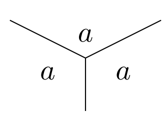
The main obstacle to showing that the depth map inverts the web map is showing that the above problem doesn't happen with a non-elliptic web. With the following lemma, the details of the proof that depth map is the inverse of the web map are easy to check.

Lemma 14. *It is not possible to have*



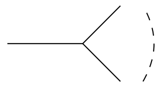
in a non-elliptic web.

Proof. Suppose that we do have

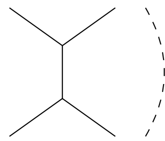


in a non-elliptic web.

Define the ***curvature*** between two endpoints of a web as $180°-60°\times(\#$ of vertices between them). E.g.



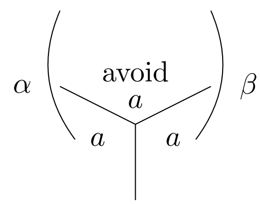
curvature $= 180° - 60° = 120°$



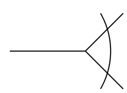
curvature $= 60°$

Observe that non-elliptic webs have $\geq 360°$ total curvature. Why? Check that tree webs have exactly $360°$ total curvature, and closing an n-gon adds $60(n-6)°$.

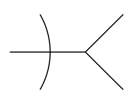
Let α, β be min cut paths to two of these faces, as close to each other as possible:



Let's consider what the total curvature along β might be, keeping in mind that α is off to the left of β somewhere. Along β,

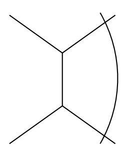


is impossible, as

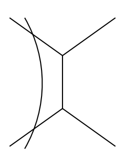


crosses fewer edges and β was chosen to be a min cut path.

Similarly,



can't occur, as

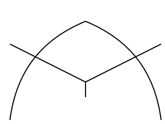


is also a min cut path but is closer to α, contradicting that we chose α and β as close as possible.

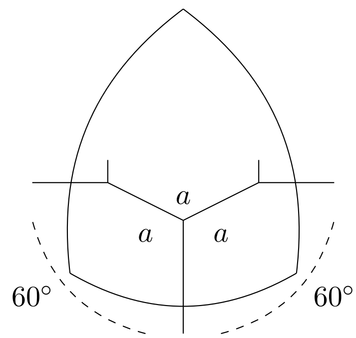
These are the only possible configurations of adjacent vertices that contribute positively to curvature, hence we conclude that the total curvature along β (and, by symmetry, along α) is $\leq 0°$.

If we join α and β to form a closed loop, then they enclose a "sub-web" of our original non-elliptic web, which itself is therefore non-elliptic. This non-elliptic web will have total curvature equal to sum of the curvatures of α and β (which as we just argued is $\leq 0°$), plus whatever curvature we gain by joining them at the top and at the bottom.

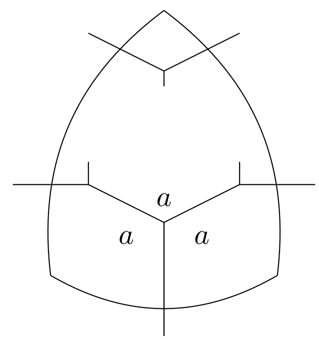
We gain at most $120°$ curvature from the top via a connection like:



At the bottom, because we cannot pass through the third face of depth a, we can pick up at most $60°$ curvature from each side:



Hence the total curvature of this enclosed web



is at most $120° + 60° + 60° = 240°$, contradicting non-elliptic. □

Kevin adds: I ended class with the assurance that the rest of the details fall into place after this lemma. There are several little things that still need checking, so here's a bit more on how we can wrap up showing that the depth and web maps are inverses. Each step requires a little straightforward work, but nothing that involves any creative new ideas (though brief hints are supplied in parentheses).

(1) Once we know the depth map is well-defined thanks to the lemma, we see that it clearly is invertible. Since we already know it's a left inverse for the web map, it suffices now to show that the depth map always results in a tripod diagram corresponding to a NCT.
(2) As long as we have a connected component with ≥ 4 edges in some tensor diagram, there will always be some internal edge between faces of the same depth (use the fact that adjacent depths differ by at most 1 to conclude that every vertex is part of two faces of the same depth). Conclude that the depth map will leave us with some sort of tripod diagram.
(3) The middle legs of the resulting tripods do not intersect anything. (Use the lemma again!)
(4) The first legs can only intersect third legs, and third legs can only intersect first legs. (Consider depths.)
(5) No two legs intersect more than once. (If they did, then the resulting tripod diagram must contain a 2-gon, though not necessarily bounded by the two legs in question. This can only result from a 2- or 4-gon in the original web, impossible since we start with non-elliptic webs.)
(6) The previous observation implies that we have drawn our tripod diagram without unnecessary crossings. The two observations before that are precisely the conditions for a tripod diagram to correspond to a NCT. Hence we have succeeded in showing that the depth map results in a tripod diagram corresponding to a NCT, which concludes the proof.

A few notes on the literature: As far as I know, Julianna Tymoczko's paper "A simple bijection between standard (n, n, n) tableaux and irreducible webs for $\mathfrak{sl}_3$" (arXiv:1005.4724) is the first appearance of what I call the "web map" (*resolving an m-diagram*, in her language). However, she doesn't phrase things in terms of NCT, so her map looks a little more involved since she effectively has to incorporate the bijection between SYT and NCT. She also relies on Greg Kuperberg's general result for rank 2 spiders to conclude that the web map is a bijection. In order to outline a self-contained proof, I have borrowed the curvature argument from Kuperberg's general proof.