NOTES FOR NOVEMBER 26, 2012: INTRODUCTION TO CRYSTALS

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1. Definition and First Examples

Crystals do not describe the representation theory of $\mathfrak{gl}_n$ but instead describe the representation theory of certain quantum groups. Still, the similarities in these structures allow us to describe certain properties of $\mathfrak{gl}_n$ in terms of crystals.

Definition. A $\mathfrak{gl}_n$ crystal consists of a set $\widetilde{B}$ with a distinguished element 0 together with the following functions. We write $\widetilde{B} = \{0\} \sqcup B$. We have a map

$$\mathrm{wt}\colon B \to \mathbb{Z}^n,$$

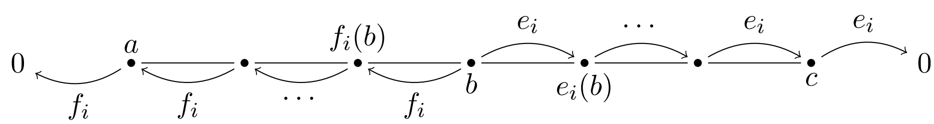
and maps

$$e_i, f_i\colon \widetilde{B} \to \widetilde{B}, \quad 1 \le i \le n-1.$$

These satisfy the following axioms:

- $e_i(0) = f_i(0) = 0$.
- If $e_i(b) \neq 0$ and $\mathrm{wt}(b) = (a_1, a_2, \dots, a_n)$, then
$$\mathrm{wt}(e_i(b)) = (a_1, a_2, \dots, a_i + 1, a_{i+1} - 1, \dots, a_n).$$
- If $f_i(b) \neq 0$ and $\mathrm{wt}(b) = (a_1, a_2, \dots, a_n)$, then
$$\mathrm{wt}(f_i(b)) = (a_1, a_2, \dots, a_i - 1, a_{i+1} + 1, \dots, a_n).$$
- If $e_i(b) \neq 0$ then $f_i(e_i(b)) = b$.
- If $f_i(b) \neq 0$ then $e_i(f_i(b)) = b$.

  Remark. Notice that if B is a finite set, then the axioms guarantee that there is a highest weight! Indeed, we have the following picture:



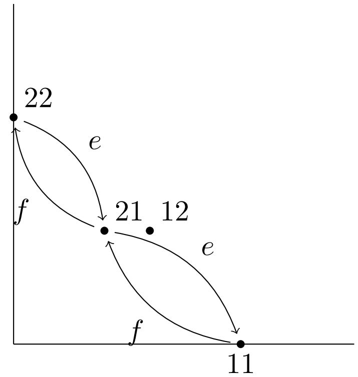
  Between each node, e_i and f_i are mutual inverses. Note that this string must be finite, and hence we must eventually get to elements wherein e_i kills the rightmost (call this c) and f_i kills the left-most (call this a). Note that this cannot have a cycle because of the way the e_i and f_i operators change the weight. We will call this the (e_i, f_i) string through b.

- For any (e_i, f_i) string, $\mathrm{wt}(a) = s_i \cdot \mathrm{wt}(c)$, where $s_i \in S_n$ acts on a string of n numbers by swapping the ith and $(i+1)$th entries. Here, a and c are as in the remark.

Example. Here is a $\mathfrak{gl}_2$ crystal. We have $B = \{11, 12, 21, 22\}$ with the data

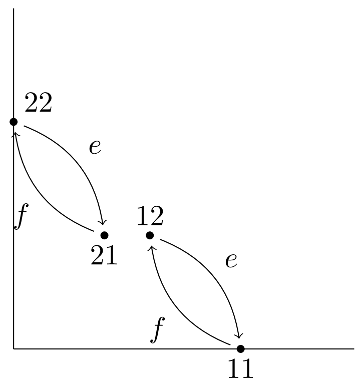
$$\mathrm{wt}(11) = (2, 0),$$
$$\mathrm{wt}(12) = (1, 1),$$
$$\mathrm{wt}(21) = (1, 1),$$
$$\mathrm{wt}(22) = (0, 2).$$

A crystal looks like



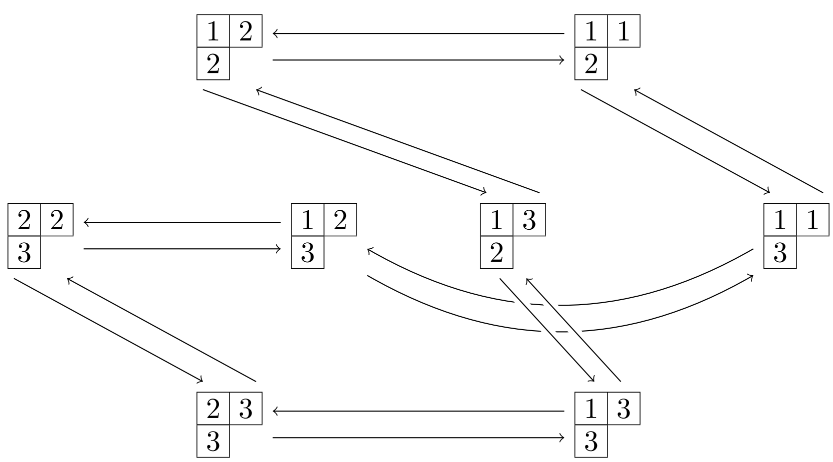
If we made the length 0 string be 21 and the length 2 string be $11 \longleftrightarrow 12 \longleftrightarrow 22$, that would also be a crystal (in fact, an isomorphic one).

An example of a non-crystal would be:



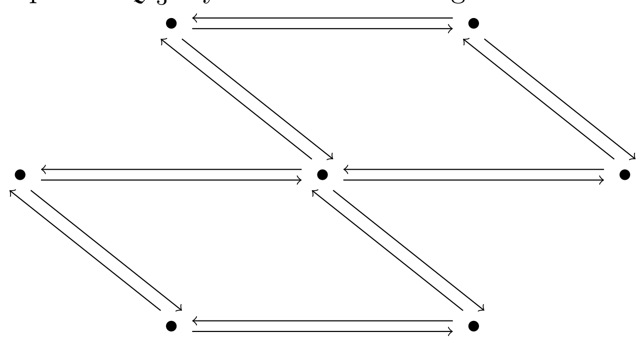
This is not a crystal because it does not satisfy the last axiom in Definition 1. $\Diamond$

Example. An example of a $\mathfrak{gl}_3$ crystal is the following:



Here $\longleftarrow$ is f_1, $\longrightarrow$ is e_1, $\searrow$ is f_2 and $\nwarrow$ is e_2. $\Diamond$

Example. Another example of a $\mathfrak{gl}_3$ crystal is the following:



However, this is only a virtual character! Explicitly, if we write down the generating function for the weight function of this crystal, it is $x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2 + xyz = s_{21}(x, y, z) - s_{111}(x, y, z)$. It is a $\mathbb{Z}$-linear combination of Schur polynomials, but the coefficients are not all positive! In other words, this is not a character of a representation. Later, we will learn to say that this is not a regular crystal. ♢

Remark. The (e_i, f_i) and (e_j, f_j) operators are only related through their interaction with the weight map $\mathrm{wt}\colon B \to \mathbb{Z}^n$. This forces that (e_i, f_i) changes the parity of the lengths of the $(e_{i\pm1}, f_{i\pm1})$ strings, and conserves the parity of the length of the (e_j, f_j) strings for $|i - j| \geq 2$. (This is a good exercise to check!) When we see regular crystals, later, we will have direct axioms about how (e_i, f_i) and (e_j, f_j) relate. ♢

Remark. This is a remark about notation in the literature. Oftentimes, people denote the number of steps to get to the final rightmost string element by $\epsilon_i(b)$ and the number of steps to get to the final leftmost string element by $\phi_i(b)$. The weight changes can be expressed in terms of some complicated linear relation on $\epsilon_i(b)$ and $\phi_i(b)$, and some people will phrase the axioms of a crystal in this language instead of in terms of the stepwise weight changes. ♢

2. An Important Example: The Word Crystal for $\mathfrak{gl}_2$

This is called *the word crystal for* $\mathfrak{gl}_2$. It is a building block for all other crystals we will discuss.

Consider $B = \{1, 2\}^n$. This will be a model for $V^{\otimes n}$, where V is the two-dimensional standard representation of $\mathfrak{gl}_2$. The weight of a word $w \in B$ is $\mathrm{wt}(w) = (\#\text{ of 1s}, \#\text{ of 2s})$.

For $n = 3$, we have the following elements of B:

222	221	211	111
	212	121	
	122	121	

This part rewritten by David. In order to build a crystal, we have to define e and f operators which organize these 8 words into three strings, one of length 3 and two of length 1. I can't fully motivate the definition, but here are some clues that might help us make peace with it.

- The weight axiom requires that e must decrease the number of 2's by one and increase the number of 1's by one. This could happen by, for example, making three 2's into 1's and making two 1's into 2's. However, it seems more natural to always make a single 2 into a 1 and this is what we will do. (Similarly, f will make a 1 into a 2.)
- We want strings of length 0 in $\{1, 2\}^{2n}$ to correspond to trivial subreps of $V^{\otimes 2n}$. The dimension of the trivial subspace in $V^{\otimes 2n}$ is the Catalan number $\frac{(2n)!}{(n+1)!n!}$. Among the things the Catalan number counts is ***ballot sequences***: Strings of n 1's and n 2's so that

every initial subword has (weakly) more 1's than 2's. For example, here are the ballot sequences of length 5:

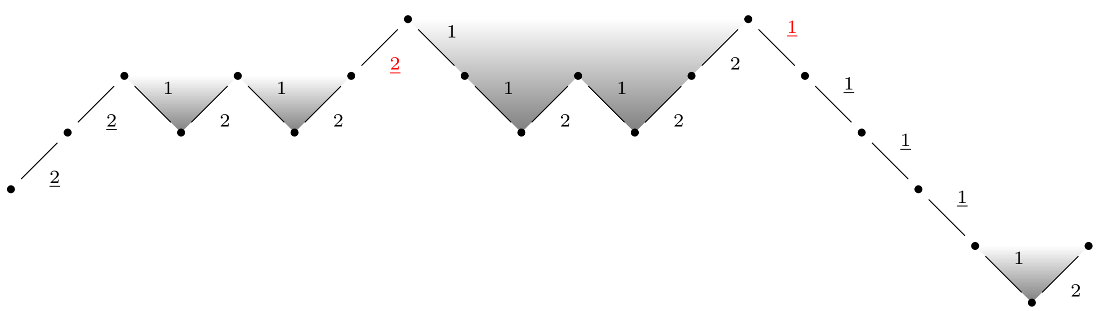
$$111222,\ 121122,\ 112212,\ 112122,\ 121212.$$

We will arrange for these to be the length zero strings of our crystal.

- The number of length ℓ strings in $V^{\otimes n}$ will be 0 if $\ell \not\equiv n \mod 2$. If $n \equiv \ell \mod 2$, then the number will be $\binom{n}{n/2-\ell/2} - \binom{n}{n/2-\ell/2-1}$. There is a standard reflection argument[1] for counting ballot sequences. That argument shows that this difference of binomial coefficients is the number of words with $(n+\ell)/2$ copies of 1 and $(n-\ell)/2$ copies of 2, such that every initial subword has more 1's than 2's. Those words will be the high weight elements.

We now give the actual rule, by showing how it operates on the sting 2212122112122111112:

Example.



The above picture represents the word 2212122112122111112. The shading is formed by "shining" light from the east and the west onto the above mountain range. Then the "lit" parts of the word are the underlined digits in $\underline{22}1212\underline{{\color{red}2}}112122\underline{{\color{red}1}}\underline{111}12$. ◊

Notice that lit numbers always look like a string of 2s followed by a string of 1s. The operator e (resp. f) changes the center lit 2 (resp. 1) to 1 (resp. 2). By the "center lit 2", we mean the boundary between where the 2s turn into 1s. These central numbers are marked in red above. The operator maps to 0 if there is no lit 2 (resp. 1) to change.

Claim. This doesn't change the lit and unlit sets.

This paragraph added by David. Once this claim is checked, it is clear that e and f are inverse as appropriate. Furthermore, the number of shadowed 1's and shadowed 2's is equal (call it s) and, by the above claim, s is preserved by the crystal operators. At the end of the $\mathfrak{gl}_2$ strings, the lit numbers are entirely 1^t and entirely 2^t. So the two ends of the string have weights $(s+t, s)$ and $(s, s+t)$, establishing the final crystal axiom.

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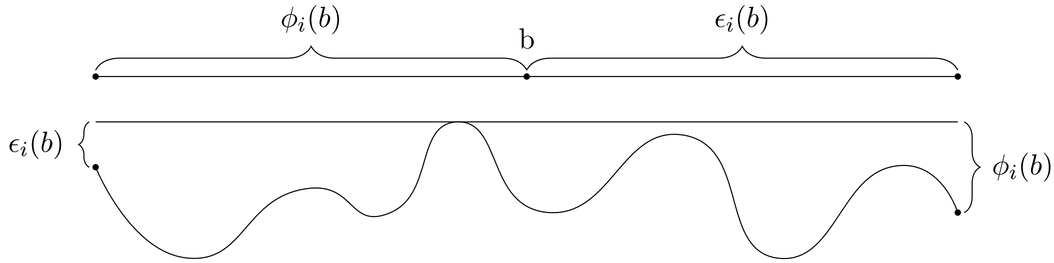
[1]See, for example, Marc Renault, *Lost (and Found) in Translation: André's Actual Method and its Application to the Generalized Ballot Problem*, Amer. Math. Monthly, April 2008.

This paragraph added by David. A helpful picture is



In other words, the distances from b to the ends of its string are given by the heights from the left and right ends of the mountain range to its highest peak(s). Notice that, in a ballot word, the mountain range is entirely a subterranean crater with its endpoints at the same height, so nothing is lit and we get a string of length 0.

This is the $\mathfrak{gl}_2$ word crystal. The $\mathfrak{gl}_n$ ***word crystal*** has $B = \{1, \ldots, n\}^d$ and (e_i, f_i) act (as above) on the $(i, i+1)$ substring leaving all other letters alone.

Next time, we will discuss how to put crystal structure on tableaux.