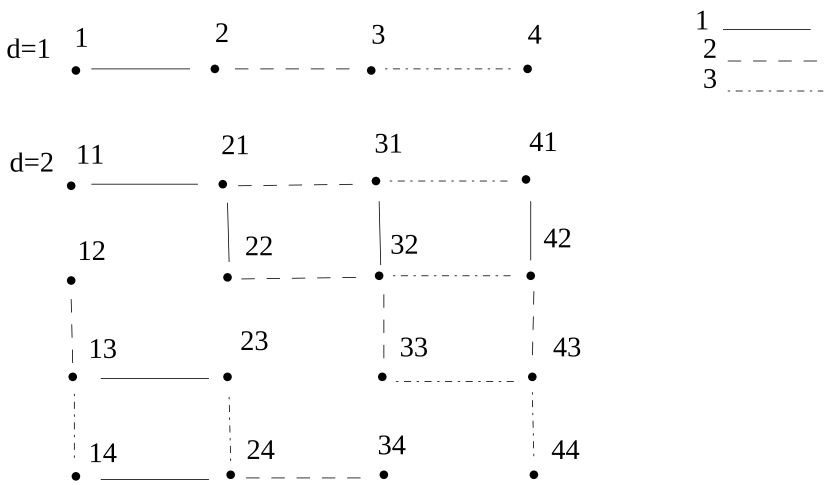
NOV 28 2012, CRYSTAL STRUCTURE ON WORDS AND TABLEAUX

YI SU

Last time we defined crystal structure on the words containing two numbers, i.e. $\{1,2\}^d$. In general , we can define the crystal structures on $\{1,2,\ldots,n\}^d$, where e_i, f_i act on $i, i+1$, and ignore everything else. The following are some examples.

Example 1. *The pictures below are the crystal structure on words* $\{1,2,3,4\}$ *when* $d = 1, 2$



As for $d = 3$*. There are four connected components. They are*

- $\{ijk|i \geq j \geq k\} \leftrightarrow Sym^3V$
- $\{ijk|i < j < k\} \leftrightarrow \wedge^3V$
- $\{ijk|i \geq j < k\} \leftrightarrow V_{(21)}$
- $\{ijk|i < j \geq k\} \leftrightarrow V_{(21)}$

Note: it is not always the case that number of connected components of crystal structure on $\{1,2,\ldots,n\}^d$ *is* 2^d*. For example, in the case* $n = 4$*,* $d = 4$*, the number of connected components is* 10 *instead of 8.*

Let λ be a partition (or ν/μ a skew shape) with $|\lambda| = d$ (or $|\nu/\mu| = d$). Embed the set of semistandard Young tableaux into the set of words $\{1,2,\ldots,n\}^{|\lambda|}$ by reading right to left, and then top to bottom:

				1	1	
		1	2	2		$\mapsto$ 112214321
1	2	3	4			

Theorem 2. *This set of words plus empty word is closed under crystal operators. If λ is a straight shape, then it is a connected component of the word crystal.*

Proof. Fix i, and consider the action of (e_i, f_i). It suffices to look at the parts contains i and $i+1$ (everything else won't change the inequality). It has to be in the following form.

									i	i
					i	i	i	$i+1$		
				i	$i+1$	$i+1$	$i+1$			
i	i									

Every vertical stack gives the reading words $iii(i+1)(i+1)(i+1)$, which will not be changed by the action of e_i, f_i. So after applying f_i, the i which is changed to $i+1$, must be at the right hand end for one of the $(i, i+1)$ row, and the box below which is not $i+1$. Similarly, after applying e_i, the $i+1$ which is changed to i, must be at the left hand end for one of the $(i, i+1)$ row, and the box above which is not i. This preserves the semistandardness of the tableau.

(Note: Any convention of reading orders work in the proof as long as we read left to right and top to bottom. For example, if we read down columns, going right to left, then a vertical stack would give $i(i+1)i(i+1)i(i+1)$ rather than $iii(i+1)(i+1)(i+1)$, but either way it would be shadowed and thus unchanged by the (e_i, f_i).)

Let λ be a straight shape partition. Let $Crys(\lambda)$ be the crystal we just built on $SSYT(\lambda, n)$. We will show that the SSYT whose ith row is filled with i's is the unique high weight element.

An element b is high weight in a crystal if $e_i(b) = 0\ \forall i$. We will show by induction on i that the ith row of a high weight element in $SSYT(\lambda, n)$ must be element i. If $i = 1$, and not all boxes are filled with 1's, then the right most box is not 1. Say it's k. Call the corresponding word b. Look at the e_k string starting at k. The first letter k must be lit, so $e_{k-1}(b) \neq 0$, a contradiction.

1	1	1	1	$\cdots$	k

Inductively, suppose we have all k's in the kth row for $1 \leq k \leq i-1$. Since the top $i-1$ rows won't affect the action of e_k for $k \geq i$, we can safely remove them, and the rest of the arguement will be exactly the same as the base case.

1	1	1	1	1	1	1	1
2	2	2	2	2			
$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$i-1$	$i-1$	$i-1$	$i-1$				
i	i	i	r				

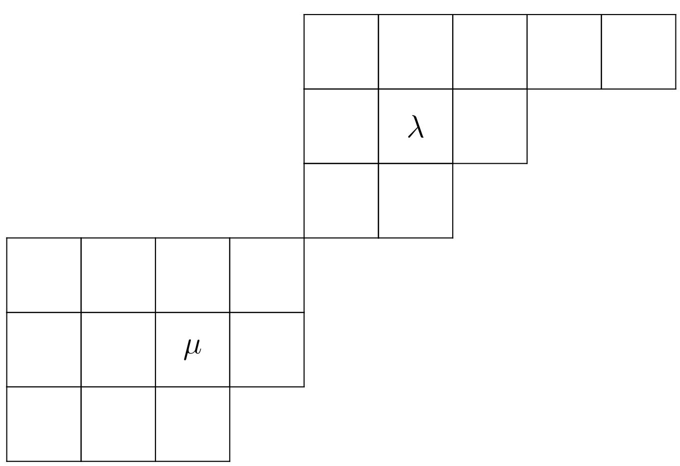
This shows that if λ is a straight shape, there is a unique high weight element in the crystal $Crys(\lambda)$, which finishes the claim

The following is what's coming up next:

Theorem 3. *Every connected component of the word crystal is isomorphic to* $Crys(\lambda)$ *for some* λ*, and the number of occurance of* $Crys(\lambda)$ *in* $\{1, 2, \ldots, n\}^{|\lambda|}$ *is the number of SYT of shape* λ.

Corollary 4. *Every connected component of* $Crys(\nu/\mu)$ *is isomorphic to* $Crys(\lambda)$ *for some* λ*, and number of copies of* $Crys(\lambda)$ *in* $Crys(\nu/\mu)$ *is the Littlewood-Richardson coefficient* $c^{\nu}_{\lambda\mu}$

On the other hand, if we would like to view LR coefficient as the coefficient of s_ν in the product $s_\lambda s_\mu$, we can define $\lambda * \mu$ to be



The number of occurance of $Crys(\nu)$ in $Crys(\lambda * \mu)$ is $c^{\nu}_{\lambda\mu}$. We can prove these in terms of high weight vectors.