NOTES FOR NOVEMBER 30: REGULAR CRYSTALS

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Problem set 11 is posted. Tried to fit in an extra lecture with the problem set!

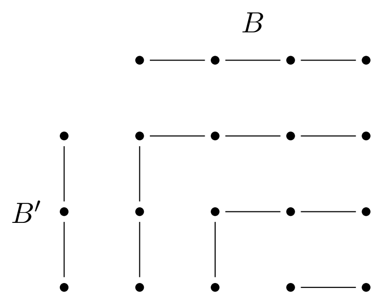
We're not going to spend much time motivating regular crystals. They were invented by John Stembridge. The arrows mean something new in these pictures. (on the side board ... coming up in axioms section).

1. Tensor product of crystals, motivating regularity

Q: How natural are these axioms?

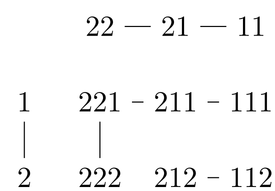
A: Let me answer a different question: How motivated is the word crystal? Because these axioms do correctly describe the word crystal.

Define a tensor product of crystals. Let $(B, \mathrm{wt}, e_i, f_i)$ and $(B', \mathrm{wt}', e'_i, f'_i)$ be crystals. We can put a crystal structure on $B \times B'$ with weight $\mathrm{wt}(b \times b') = \mathrm{wt}(b) + \mathrm{wt}'(b')$. The rule is given pictorially.



$e\ (\rightarrow), f\ (\leftarrow), e'\ (\uparrow), f'\ (\downarrow)$

For word crystals, concatenation produces the tensor product. E.g.

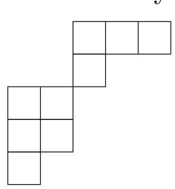
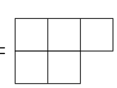
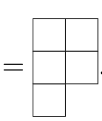


Using tensor products of crystals, there is an alternate definition for the word crystal. Alternate definition: The word crystal is

$$\{1, 2, \ldots, n\} \otimes \{1, 2, \ldots, n\} \otimes \cdots \otimes \{1, 2, \ldots, n\}.$$

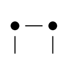
However, this definition is computationally horrible.

For tableaux, the tensor product is $\mathrm{crys}(\lambda) \otimes \mathrm{crys}(\mu) \cong \mathrm{crys}(\lambda * \mu)$, with (e.g.)

$\lambda * \mu =$  where $\lambda =$  $\mu =$ .

2. Axioms

For $|i - j| \geq 2$

 implies  $(\times 4$ symmetry).

These axioms about (e_i, f_i) and (e_j, f_j) mean that if $|i - j| \geq 2$, then (e_i, f_i) and (e_j, f_j) commute.

Consequence: If $e_i b$ and $e_j b \neq 0$, then $e_i e_j b = e_j e_i b \neq 0$. To see this look at the picture

$$\begin{array}{ccc} e_i b & — & b \\ | & & | \\ 0 & — & e_j b \end{array}$$

so that

$$e_i f_j(e_j b) = f_j e_i(e_j b)$$

which is a contradiction.

The interesting part is when they are adjacent, i.e. $|i - j| = 1$.

Relation between (e_i, f_i) and (e_{i+1}, f_{i+1}):

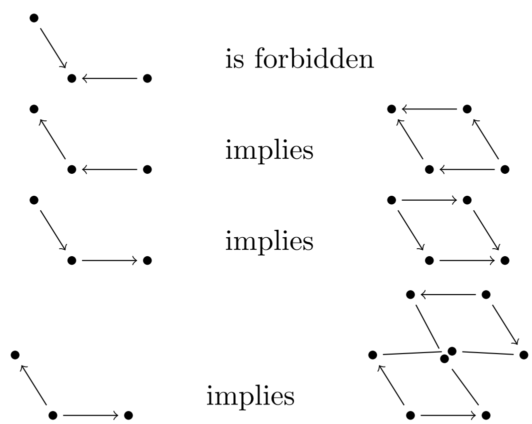
- (This is an axiom that arrows exist.)
  If $e_{i+1}b \neq 0$, then length of i-string through b is the (length of the i-string through $e_{i+1}b) \pm 1$.
  Likewise for switching i and $i+1$, and for switching e and f.

  Decorate the picture with arrow heads to see which of the two situations holds. The arrow goes from the shorter string to the longer one.
  Note that an i edge may be $\longrightarrow$ for $(i, i+1)$ and $\longleftarrow$ for $(i-1, i)$, that is, the arrow direction depends on which strings we're comparing.

Now that we know what the arrows mean, we can explain these pictures (which have been up on the side-board).

For $|i - j| = 1$



(and also 180° rotations of all of these).

For the last one, no arrow heads means that I don't know which way it goes.

In words, if $f_i b, f_{i+1}b \neq 0$ and f_i, f_{i+1} lengthens, then

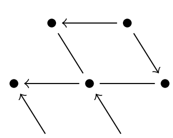
$$f_i f_{i+1}^2 f_i b = f_{i+1} f_i^2 f_{i+1} b$$

and

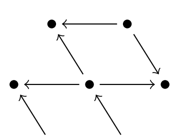
$$f_{i+1}^2 f_i \xrightarrow{f_i} f_i f_{i+1}^2 f_i b \qquad \text{and} \qquad f_i^2 f_{i+1} \xrightarrow{f_{i+1}} f_{i+1} f_i^2 f_{i+1} b$$

are shortening.

We can't have



because



self-perpetuates. So the middle two dots can't collide.

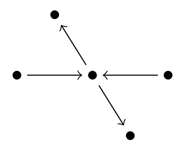
3. Playing with axioms

They are very constraining, even when excluding the last weird one.

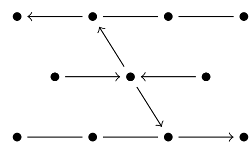
The great thing about these axioms is that they are local. To state the axioms, however, we need to talk about the string length.

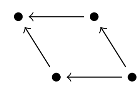
We'll play with these axioms, and show some consequences.

It turns out that $\longrightarrow \longleftarrow$ is fobidden. This is because $\longrightarrow \longleftarrow$ implies

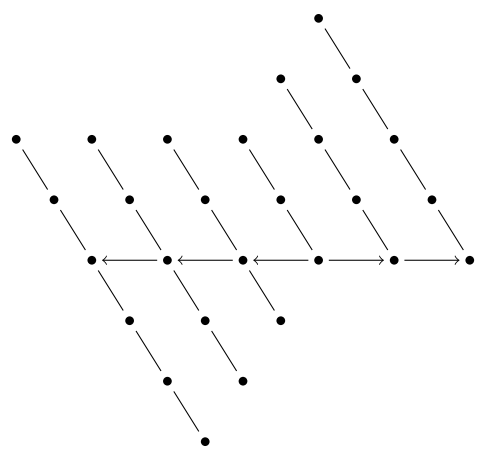


where the two vertical arrow are lengthening. This implies



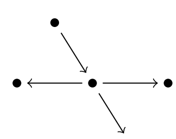
which gets a contradiction, becase we have to have  , which is backwards from an arrow we already have.

Next example: We can get a bow-tie-like shape



from shortening and lengthening.

Another thing. This



is also forbbiden, so we instead get the picture in the handouts.

4. Plan for other classes

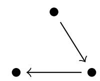
Monday

- Every regular connected crystal has exactly one highest element.
- If B and B' are connected regular crystals, with high weight elements u and u' which have the same weight $\mathrm{wt}(u) = \mathrm{wt}(u')$, then $B \cong B'$.

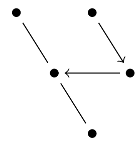
Wednesday: The hard crystal is regular. The proof is hard. Consequences: Every tableaux crystal is regular. And $\mathrm{crys}(\lambda)$ is the unique regular connected crystal with high weight λ.

5. Note added from after class discussion (by David)

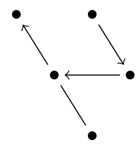
Kevin Carde pointed out that



is forbidden by the axioms. To see this, notice that the horizontal arrow forces the existence of the two dots shown below



Since e_i and e_{i+1} cannot be directed into the same vertex, we must have



But now we must have a commuting parallelogram in the upper-left, contradicting the supposed downward arrow.

From this rule, it is easy to deduce several of the other rules mentioned in class. Since Kevin pointed it out to me, I've found that it simplified a lot of arguments about what can and cannot happen in a regular crystal.