SCHUR WEYL DUALITY

In this lecture, we draw connection between irreducible representations of S_d and irreducible representations of GL_n. Let $Sp(\lambda)$ be the Specht module, which is $(1, 1, \ldots, 1)$ weight space of of V_λ, where $|\lambda| = d$.

Theorem 1 (Schur Weyl Duality). *Let* $\dim V = n$. *Then* $V^{\otimes d} \cong \bigoplus_{|\lambda|=d} Sp(\lambda) \otimes V_\lambda(n)$ *as an* $S_d \times GL_n$ *representation.*

Proof. Inside $\mathbb{C}[z_{ij}]_{\substack{1\le i\le d\\ 1\le j\le n}}$ look at terms of degree 1 in each row. Since S_d switches rows, and GL_n acts on rows, one knows that this space is isomorphic to $V^{\otimes d}$. On the other hand, $\mathbb{C}[z_{ij}] \cong \bigoplus_\lambda V_\lambda(d)^\vee \otimes V_\lambda(n)$. Terms of degree 1 in each row of (z_{ij}) corresponds to pick $(1, 1, \ldots, 1)$ weight space from $V_\lambda(d)^\vee$, which is $Sp(\lambda)^\vee$, where $|\lambda| = n$. Therefore, comparing two sides and notice representation of symmetric groups are self-dual, we get

$$V^{\otimes d} \cong \bigoplus_{|\lambda|=d} Sp(\lambda)^\vee \otimes V_\lambda(n) \cong \bigoplus_{|\lambda|=d} Sp(\lambda) \otimes V_\lambda(n)$$

□

Now we see Young Symmetrizer from the Schur Weyl Duality point of view. Let $|\lambda| = d$, $a_\lambda = \frac{1}{|S_\lambda|}\sum_{\sigma\in S_\lambda}\sigma$, and $b_\lambda = \frac{1}{|S_\lambda|}(-1)^\sigma\sigma$. According to the problem set, we know $a_\lambda V^{\otimes d} = \otimes_k Sym^{\lambda_k} V$. As we have computed before, this is $V_\lambda \oplus (\oplus_{\mu\prec\lambda} V_\mu^{\otimes n_\mu})$. Similarly $b_{\lambda^T} V^{\otimes d} = \otimes_k \wedge^{\lambda_k^T} V = V_\lambda \oplus (\oplus_{\mu\succ\lambda}) V_\mu^{n_\mu}$. Hence, a_λ acts on $Sp(\lambda)$ by rank 1 map, and acts by 0 map on $Sp(\mu)$ for $\mu \succ \lambda$ and b_{λ^T} acts on $Sp(\lambda)$ by a rank 1 map and acts by 0 map on $Sp(\mu)$ for $\mu \prec \lambda$.

In particular, $a_\lambda b_{\lambda^T}\mathbb{C}[S_d] \cong Sp(\lambda)$ as an S_d representation. (Here $a_\lambda b_{\lambda^T}\mathbb{C}[S_d]$ is an S_d representation by the right S_d-action.)

In particular, we can now prove what we observed last time:

Proposition 2. $Sp(\lambda^T) \cong Sp(\lambda) \otimes Sgn$, *where* Sgn *is the sign representation.*

Proof. Define the map $\epsilon : \mathbb{C}[S_d] \longrightarrow \mathbb{C}[S_d]$ by $\epsilon(\sigma) = (-1)^\sigma\sigma$ for $\sigma \in S_d$, extended by linearity. Note that this is a map of rings. Clearly, $\epsilon(a_\lambda) = b_\lambda, \epsilon(b_\lambda) = a_\lambda$. So $\epsilon(a_\lambda b_{\lambda^T}\mathbb{C}[S_d]) = b_\lambda a_{\lambda^T}\mathbb{C}[S_d]$. On the other hand, $\mathbb{C}[S_d] \cong \oplus_{|\lambda|=d} Sp(\lambda) \otimes Sp(\lambda)$. By the observation above, $a_\lambda b_{\lambda^T}\mathbb{C}[S_d] \cong b_{\lambda^T} a_\lambda \mathbb{C}[S_d] \cong Sp(\lambda)$. Hence $\epsilon(Sp(\lambda)) = \epsilon(a_\lambda b_{\lambda^T}\mathbb{C}(S_d)) = b_\lambda a_{\lambda^T}\mathbb{C}[S_d] = Sp(\lambda^T)$. Also notice the map ϵ maps $Sp(\lambda)$ to $Sp(\lambda) \otimes Sgn$. Therefore, $Sp(\lambda^T) \cong Sp(\lambda) \otimes Sgn$. □

Schur-Weyl duality can often be used to make constructions "natural", in a not very useful way. For example, we have seen the map $\omega : \Lambda \to \Lambda$, which takes s_λ to s_{λ^T}. Is there a "natural" map on GL_n-representations which realizes it? Why, yes!

One useful but not insightful application: Fix $d \le n$, V is standard representation.

Let $\mathcal{C}$ be the category of polynomial GL_n representations, where tId acts by t^d. Then we define a functor $\mathcal{C} \to \mathcal{C}$ by:

$$W \longrightarrow Hom_{S_d}(Hom_{GL_n}(W, V^{\otimes d}) \otimes Sgn, V^{\otimes d}).$$

Thus functor takes representations with character f to representations with character $\omega(f)$.

In general, Schur-Weyl duality is an equivalence of categories between

$$\{\text{polynomial representations of } GL_n \text{ on which } t \cdot Id \text{ acts by } t^d\}$$

and

$$\{S_d \text{ representations}\}$$

The maps are $W \mapsto \mathrm{Hom}_{GL(V)}(V^{\otimes d}, W)$ and $W \mapsto \mathrm{Hom}_{S_d}(V^{\otimes d}, W)$.

So, if you have a construction which you can categorify on the S_d side, this will let you categorify it on the GL_n side.