APPLICATIONS OF SCHUR-WEYL DUALITY:

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Last time we proved ***Schur-Weyl Duality***:

$$V^{\otimes d} \cong \bigoplus_{|\lambda|=d} Sp(\lambda) \otimes V_\lambda(n)$$

as a $S_d \times \mathrm{GL}(V)$-representation, where ($n = \dim V$).

Let's see this theorem in action.

1. Application 1: The First Fundamental Theorem of Invariant Theory

We'll finally prove that the natural map $\mathbb{C}[S_d] \to \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes d})$ is always sujrective. From Problem Set 6, we know it's an isomorphism in the case $n \geq d$.

Proof. From the decomposition given by Schur-Weyl duality, thinking one isotypic component at a time,

$$\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes d}) = \bigoplus_{V_\lambda(n) \neq 0, |\lambda|=d} \mathrm{End}(Sp(\lambda)).$$

What we are doing here is noting that all of the nonzero $V_\lambda(n)$ have 1-dimensional $\mathrm{GL}(V)$-endomorphism rings, and that $\mathrm{GL}(V)$ acts trivially on the $Sp(\lambda)$ factors (so these stick along for the ride).

Note it *is* possible that $V_\lambda(n) = 0$, but $Sp(\lambda) \neq 0$; these don't show up on the RHS.

At this point, we are done by Mashcke's Theorem (which we mentioned in the October 5, Consequences of Peter-Weyl lecture): the group algebra spans

$$\bigoplus \mathrm{End}(V_i)$$

as the V_i vary over the G-irreps, each taken exactly once.

Hence in our case, $\mathbb{C}[S_d] \to \bigoplus \mathrm{End}(Sp(\lambda))$ is surjective (not every λ has to show up here, but each one that does shows up at most once, so we are good by Maschke). □

By the tensor-hom adjunction, we know that

$$\begin{aligned} \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes d}) &:= \mathrm{Hom}_{\mathrm{GL}(V)}(V^{\otimes d}, V^{\otimes d}) \\ &= \mathrm{Hom}_{\mathrm{GL}(V)}(V^{\otimes d}, \mathrm{Hom}((V^\vee)^{\otimes d}, \mathbb{C})) \\ &= \mathrm{Hom}_{\mathrm{GL}(V)}((V^\vee)^{\otimes d} \otimes V^{\otimes d}, \mathbb{C}). \end{aligned}$$

This last way of writing $\mathrm{End}(V^{\otimes d})$ is often useful. Let's see how $\mathbb{C}[S_d]$ shows up concretely in this case. Given a simple tensor $u_1 \otimes \cdots u_d \otimes v_1 \otimes \cdots v_d$ with the $u_i \in V^\vee$, $v_i \in V$, and given $\sigma \in S_d$, σ gets sent to the $\mathrm{GL}(V)$-equivariant map

$$u_1 \otimes \cdots u_d \otimes v_1 \otimes \cdots \otimes v_d \mapsto \langle u_1, v_{\sigma(1)}\rangle \langle u_2, v_{\sigma(2)}\rangle \cdots \langle u_d, v_{\sigma(d)}\rangle.$$

The surjectivity we have just proven says that these types of functions generate $\mathrm{Hom}_{\mathrm{GL}(V)}((V^\vee)^{\otimes d} \bigotimes V^{\otimes d}, \mathbb{C})$.

We know from previous lectures that $V_\lambda(n)$ is nonzero if and only if $\ell(\lambda) \leq n$.

So we can rewrite our Application 1 above as

$$\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes d}) \cong \bigoplus_{|\lambda|=d, \ell(\lambda) \leq n} \mathrm{End}(Sp(\lambda)).$$

Date: November 9, 2012.

Let's look at the case $n = 2$. In this case, the dimension of the left hand side of the above is the d^{th} Catalan number, which is the number of SYT on (d, d). The right hand side has dimension given by

$$\sum_{|\lambda|=d, \ell(\lambda)\leq 2} (\#\text{ of SYT on } \lambda)^2.$$

It is a nice exercise to verify why these two numbers are the same. The hint David provided in class was to split the diagram for $\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline * & * & * & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & * & * & * \\ \hline\end{array}$ into $\begin{array}{|c|c|c|c|c|}\hline * & * & * & * & * \\ \hline * & * & * & \multicolumn{2}{c}{} \\ \cline{1-3}\end{array}$ and $\begin{array}{ccccc}\cline{3-5} \multicolumn{2}{c|}{} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} & \multicolumn{1}{c|}{*} \\ \hline\end{array}$. The algebra $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes d})$ when $n = 2$ is called the ***Temperley-Lieb Algebra***, and is a hot topic, because it is an excellent toy example of a ***planar algebra***.

2. Application 2: Littlewood-Richardson Coefficients

We define $c^{\nu}_{\lambda\mu}$ by the equality

$$s_\lambda(z)s_\mu(z) = \sum_{\nu} c^{\nu}_{\lambda\mu} s_\nu(z).$$

We also can define

$$s_\nu(x, y) = \sum_{\lambda,\mu} d^{\lambda\mu}_{\nu} s_\lambda(x)s_\mu(y).$$

The theorem is that $c^{\nu}_{\lambda\mu} = d^{\lambda\mu}_{\nu}$.

Proof. We will see the theorem by noting that both of these numbers are the coefficient of s_μ inside $s_{\nu/\lambda}$, *i.e.* $\langle s_{\nu/\lambda}(y), s_\mu(y)\rangle$.

The schur function $s_\nu(x, y)$ is defined to be

$$s_\nu(x, y) = \sum_{T\in\ \mathrm{SSYT}(\nu)} x^T y^T.$$

This is symmetric in the x and y variables, so we can focus on tableaux in which all of the "y" content happens outside the "x" content (this is like thinking the y variables are "bigger" than the x variables). Thinking this way shows the equality

$$\begin{aligned} s_\nu(x, y) &= \sum_{\lambda\subset\nu,\, T\in SSYT(\lambda),\, U\in SSYT(\nu/\lambda)} x^T y^U \\ &= \sum_{\lambda} s_\lambda(x)s_{\nu/\lambda}(y). \end{aligned}$$

Now, once we expand $s_{\nu/\lambda}(y) = \sum_\mu e_{\nu,\mu,\lambda}s_\mu(y)$ and plug in to the above, we get the equality

$$s_\nu(x, y) = \sum_{\lambda,\mu} e_{\nu,\mu,\lambda}s_\lambda(x)s_\mu(y),$$

from which we deduce that

$$e_{\nu,\mu,\lambda} =< s_{\nu/\lambda}(y), s_\mu(y) >= d^{\lambda,\mu}_{\nu}.$$

Now the adjointness property of "skewing by λ" from the September 26 lecture, states that for any $f \in \Lambda$

$$< s_{\nu/\lambda,f} >=< s_\nu, f s_\lambda > .$$

Applying this to $f = s_\mu$, we see

$$< s_{\nu/\lambda}, s_\mu >=< s_\nu, s_\mu s_\lambda >= c^{\nu}_{\lambda,\mu}.$$

This shows

$$c^{\nu}_{\lambda,\mu} = d^{\lambda,\mu}_{\nu} =< s_{\nu/\lambda}, s_\mu >$$

as claimed. □

Now let's define $C^\nu_{\lambda,\mu} = \mathrm{Hom}_{\mathrm{GL}(V)}(V_\nu, V_\lambda \otimes V_\mu)$ for $\dim V \geq \ell(\nu)$; this is a space whose dimension is $c^\nu_{\lambda,\mu}$.

Similarly let's set

$$D^{\lambda,\mu}_\nu = \mathrm{Hom}_{\mathrm{GL}_k \times \mathrm{GL}_{n-k}}(V_\lambda(k) \otimes V_\mu(n-k), V_\nu(n)).$$

We need $k \geq \ell(\lambda)$, $(n-k) \geq \ell(\mu)$ here. This space has dimension $d^{\lambda,\mu}_\nu$.

Note! The spaces $C^\nu_{\lambda,\mu}, D^{\lambda,\mu}_\nu$ are "just" vector spaces; taking Hom of two GL-reps contracts the GL action to a trivial one. David talked some more about these spaces on the November 12 lecture.

Let $\ell = |\lambda|, m = |\mu|$; we relate both of $D^{\lambda,\mu}_\nu$ and $C^\nu_{\lambda,\mu}$ to the space

$$E^{\lambda,\mu}_\nu = \mathrm{Hom}_{S_\ell \times S_m}(Sp(\lambda) \otimes Sp(\mu), Sp(\nu)).$$

This should work in general, but for our construction, we will focus on the case that $k = \ell$, $n - k = m$. (See the appendix for more on this.) First, we'll relate $D^{\lambda,\mu}_\nu$ and $E^{\lambda,\mu}_\nu$. Look at the $(1, \dots, 1)$ weight space in $V_\nu(n)$. As an S_n-rep, it is $Sp(\nu)$. Therefore, as an $S_l \times S_m$-rep, it is $Sp(\nu)|_{S_\ell \times S_m}$. Notice that $S_\ell \times S_m \subset \mathrm{GL}_\ell \times \mathrm{GL}_m$, so we can also restrict down to $S_\ell \times S_m$ by first restricting from GL_n to $\mathrm{GL}_\ell \times \mathrm{GL}_m$, and then restrict to $S_\ell \times S_m$.

From our character computations, restricting to $\mathrm{GL}_\ell \times \mathrm{GL}_m = \mathrm{GL}_k \times \mathrm{GL}_{n-k}$ first, we get

$$V_\nu(n)|_{\mathrm{GL}_k \times \mathrm{GL}_{n-k}} \cong \bigoplus_{\lambda,\mu} D^{\lambda,\mu}_\nu \otimes V_\lambda(k) \otimes V_\mu(n-k).$$

Now passing to $(1, \dots, 1)$-weight spaces,

$$Sp(\nu)|_{S_k \times S_{n-k}} \cong \bigoplus D^{\lambda\mu}_\nu \otimes Sp(\lambda) \otimes Sp(\mu),$$

from which we conclude

$$E^{\lambda\mu}_\nu = D^{\lambda\mu}_\nu.$$

Now we want to involve $C^\nu_{\lambda,\mu}$ in the picture. Look at $V^{\otimes(\ell+m)} = \bigoplus_{|\nu|=\ell+m} Sp(\nu) \otimes V_\nu$ as a $S_{\ell+m} \times \mathrm{GL}(V)$-rep. Restricting to the product of smaller symmetric groups gives

$$V^{\otimes(\ell+m)} = \bigoplus_{|\nu|=\ell+m, |\lambda|=\ell, |\mu|=m} E^{\lambda\mu}_\nu \otimes Sp(\lambda) \otimes Sp(\mu) \otimes V_\nu \tag{1}$$

as an $(S_\ell \times S_m \times \mathrm{GL}(V))$-representation.

Also, Schur-Weyl applied to $V^{\otimes \ell} \otimes V^{\otimes m}$ gives

$$V^{\otimes \ell} \otimes V^{\otimes m} = \bigoplus_{|\lambda|=\ell} Sp(\lambda) \otimes V_\lambda \bigotimes \bigoplus_{|\mu|=m} Sp(\mu) \otimes V_\mu$$

as $S_\ell \times \mathrm{GL}(V) \times S_m \times \mathrm{GL}(V)$ reps, and rearranging tensor factors gives

$$\begin{aligned} V^{\otimes \ell} \otimes V^{\otimes m} &= \bigoplus_{|\lambda|=\ell, |\mu|=m} Sp(\lambda) \otimes Sp(\mu) \bigotimes \bigoplus_{|\lambda|=\ell, |\mu|=m} V_\lambda \otimes V_\mu \\ &= \bigoplus_{|\lambda|=\ell, |\mu|=m} Sp(\lambda) \otimes Sp(\mu) \bigotimes \bigoplus_{|\nu|=\ell+m} C^\nu_{\lambda,\mu} \otimes V_\nu, \end{aligned}$$

where the last step is justified by the definition of $c^\nu_{\lambda,\mu}$ as structure constants for multiplication.

Comparing (1) and the RHS of the preceding, we read off that $C^\nu_{\mu\lambda} = E^{\lambda\mu}_\nu$.

3. Some Context: Bialgebras

We would like to provide some context for these results: You should think of $d^{\lambda\mu}_\nu$ as structure constants for comultiplication

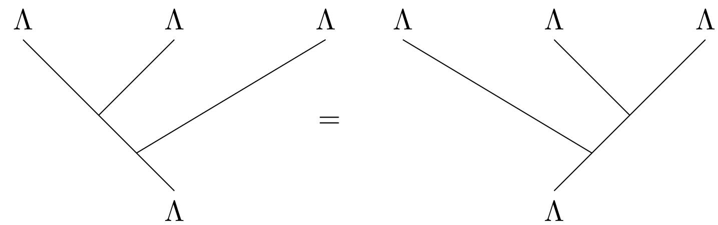
$$\Delta \colon \Lambda \to \Lambda \otimes \Lambda$$

via $f(x) \mapsto f(x, y)$, sending a symmetric function f to a function that is symmetric in "two" infinite sets of variables, x, y.

Similarly, we let $m \colon \Lambda \otimes \Lambda \to \Lambda$ by $f \otimes g \mapsto fg$ be the multiplication map; $c^\nu_{\lambda\mu}$ are the structure constants for this multiplication map.

Our multiplication and comultiplication satisfy the axioms for what is called a ***bialgebra structure***. The axioms for a bialgebra are as follows:

- The map m makes Λ into an algebra. We definitely know m is distributive (since we have defined it as a map from the tensor product). What we need to check is that it is associative; we choose to express this by the following diagram:



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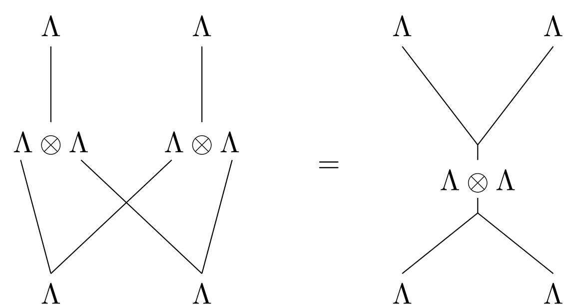
where when two lines meet, you multiply.

An algebra also needs a unit map, $\mathrm{id}\colon \mathbb{Z} \to \Lambda$ which gives isomorphisms

$$\mathbb{Z} \otimes \Lambda \xrightarrow{\mathrm{id} \otimes \mathrm{id}_\Lambda} \Lambda \xleftarrow{\mathrm{id}_\Lambda \otimes \mathrm{id}} \Lambda \otimes \mathbb{Z},$$

where both of the arrows above are isomorphisms. We could write this as a diagram if we liked.
- The bulleted point above describes the axioms for an algebra. A ***Co-algebra*** comes with a counit $\epsilon\colon \Lambda \to \mathbb{Z}$ and a comultiplication $\Delta\colon \Lambda \to \Lambda \otimes \Lambda$. These maps lead to a coalgera structure if they satisfy the same diagrams as we drew for an algebra, with the arrows reversed.
- In terms of these properties, a bialgebra is both an algebra and a coalgebra, and these structures are compatible. Compatibility means that Δ is a map of algebras, which is expressed in the diagram



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- A final object you may have heard of is a ***Hopf algebra***. A Hopf algebra is a bialgebra, but has additional structure that axiomatizes the involution ω on symmetric functions. A graded bi-algebra, such as Λ, is automatically a Hopf algebra. See Sweedler's book *Hopf Algebras* (1969) or `http://sbseminar.wordpress.com/2011/07/07/why-graded-bi-algebras-have-antipodes/`

The reason we bring all of this up, is the following beautiful result of Zelevinsky:

> Suppose R is a graded Hopf-algebra, with $R_0 = \mathbb{Z}$, which is a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis, $\{s_i \colon i \in I\}$, and suppose that the structure constants of m and Δ with respect to this basis are the same as each other and are nonnegative. Then $R \cong \Lambda^{\otimes m}$ for some choice of m, and your basis is $s_{\lambda_1} \otimes s_{\lambda_2} \otimes \cdots \otimes s_{\lambda_m}$.

This is the main result of Chapter 1 of *Representations of Finite Classical Groups: A Hopf Algebra Approach*, A. Zelevinsky (1981).

Some would say this is a philosophical explanation for why Λ shows up in so many seemingly unrelated places in math. For example, you may have heard that the cohomology ring of the Grassmannian is intimately related to Λ. One "explanation" for this is that the Grassmannian has natural geometric operations which give multiplication, co-multiplication, et cetera. For example, multiplication corresponds to intersection. From the perspective of Zelevinsky's result, once we know this, we know the cohomology ring of the Grassmannian "has" to be related to the ring of symmetric functions.

Appendix: Removing the dependence of D on $(k, n-k)$: A start

Let $|\lambda| = \ell$, $|\mu| = m$ and $|\nu| = \ell + m$. We defined

$$D_\nu^{\lambda\mu} = \mathrm{Hom}_{GL_k \times GL_{n-k}}(V_\lambda(k) \otimes V_\mu(n-k),\ V_\nu(n)|_{GL_k \times GL_{n-k}}).$$

for $k \geq \ell$ and $n - k \geq m$.

This definition apparently depends on k and n, so write it as $D^{\lambda\mu}\nu(k, n-k)$. In this appendix, we will construct natural maps $D_\nu^{\lambda\mu}(k, n-k+1) \cong D_\nu^{\lambda\mu}(k, n-k)$ and $D_\nu^{\lambda\mu}(k+1, n-k) \to D_\nu^{\lambda\mu}(k, n-k)$.

Our definition of $C_{\lambda\mu}^\nu$ also had an apparent dependence on an auxiliary n, but we constructed an isomorphism $C_{\lambda\mu}^\nu(n) \cong E_\nu^{\lambda\mu}$ for every $n \geq \ell + m$, so composing two such isomorphisms identifies $C_{\lambda\mu}^\nu(n)$ and $C_{\lambda\mu}^\nu(n')$ for different n and n'. When we identified D with E, we only did it for $k = \ell$ and $n - k = m$, so we can't use that approach to identify the different D's with each other.

Lemma: Let $n \geq |\kappa|$. Let $V_\nu(n+1)^0$ be the part of $V_\nu(n+1)$ where $\mathrm{diag}(1, 1, \ldots, 1, t)$ acts by weight t^0. Then $V_\nu(n+1)^0 \cong V_\nu(n)$ as a GL_n representation.

Proof: This is the Schur function identity

$$s_\nu(x_1, x_2, \ldots, x_n, t) = s_\nu(x_1, x_2, \ldots, x_n) + (\text{terms divisible by } t). \quad \square$$

Let $k \geq \ell$ and $n - k \geq m$. Let ϕ lie in $D_\nu^{\lambda\mu}(k, n-k+1) = \mathrm{Hom}_{GL_k \times GL_{n-k}}(V_\lambda(k) \otimes V_\mu(n-k+1),\ V_\nu(n+1)|_{GL_k \times GL_{n-k+1}})$. Restrict ϕ to $V_\lambda(k) \otimes V_\mu(n-k+1)^0$ inside $V_\lambda(k) \otimes V_\mu(n-k+1)$. We claim that the restriction of ϕ lands in $V_\nu(n+1)^0$. **Proof:** Since ϕ is a map of GL_{n+1} modules, it preserves weight spaces. In particular, ϕ preserves the property of $\mathrm{diag}(1, 1, \ldots, 1) \times \mathrm{diag}(1, 1, \ldots, 1, t)$ acting by weight t^0. So $\phi(V_\lambda(k) \otimes V_\mu(n-k+1)^0) \subseteq V_\nu(n+1)^0$.

So the restriction of ϕ gives a map

$$V_\lambda(k) \otimes V_\mu(n-k) \cong V_\lambda(k) \otimes V_\mu(n-k+1)^0 \to V_\nu(n+1)^0 \cong V_\nu(n).$$

In other words, the restriction of ϕ lies in $D_\nu^{\lambda\mu}(k, n-k)$.

We have built a map $D_\nu^{\lambda\mu}(k, n-k+1) \to D_\nu^{\lambda\mu}(k, n-k)$, sending ϕ to its restriction. We can similarly build a map $D_\nu^{\lambda\mu}(k+1, n-k) \to D_\nu^{\lambda\mu}(k, n-k)$.

Proving that this maps are isomorphisms will wait for another day – possibly November 19.