LECTURE 12: REPRESENTATION OF COMPACT GROUP AND ITS CHARACTERS

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In this lecture, G is a compact group, and all representations are continuous unless specified otherwise. In the previous lecture, we showed that every representation is a direct sum of simple representations.

Lemma 1 (Schur's Lemma)**.** *If V is a simple representation of group G over $\mathbb{C}$, then $End_G(V) = \mathbb{C} \cdot Id$.*

Proof. Let $A : V \to V$ commute with G action. Over $\mathbb{C}$, the matrix A has eigenvalue λ, so $\ker(A - \lambda \cdot Id) \neq 0$. On the other hand $\ker(A - \lambda \cdot Id)$ is a subrepresentation of V, so it is V. Thus, $A = \lambda \cdot Id$ □

The proof of lemma also gives an algorithm for decomposing a representation into subrepresentations. If the representation is given in some explicit way, then computing $End_G(V)$ is a matter of linear algebra. Generate a random element of $End_G(V)$; its eigenspaces are subreps of V.

Proposition 2. *If V and W are simple G-representations, and $V \ncong W$, then $Hom_G(V, W) = 0$.*

Proof. We prove the contrapositive statement: if there is a nonzero G-commuting homomorphism from V to W, then $V \cong W$.

Let $A : V \to W$ be such nonzero homomorphism. The subrepresentation $\ker(A)$ of V does not equal to V, so $\ker(A) = 0$. Similarly the subrepresentation $\text{Im}(A)$ of W does not equal to 0, so $\text{Im}(A) = W$. Therefore, A is both injective and surjective, thus a bijection. □

Following the above results, let U_i be a list pairwise nonisomorphic simple representations, and $V \cong \oplus U_i^{\oplus a_i}$, $W \cong \oplus U_i^{\oplus b_i}$. Then $Hom_G(V, W) = \oplus_i Mat_{a_i \times b_i}\mathbb{C}$. Also, if U is a simple representation, and V is any representation, then $V \cong W_1 \oplus W_2$, where $W_1 \cong U^{\otimes a}$ and W_2 has no subrepresentation isomorphic to U. In another word, U isotypic component of V is the image of $Hom(U, V) \otimes U \to V$.

Corollary 3. *If $\oplus_i U_i^{\oplus a_i} \cong \oplus_i U_i^{\oplus b_i}$, then $a_i = b_i$.*

Proof. Isomorphism means that we have inverse maps g, h between the spaces. Say $g = (g_1, \ldots, g_n)$, where $g_i \in Mat_{a_i \times b_i}\mathbb{C}$, and $h = (h_1, \ldots, h_n)$, where $h_i \in Mat_{b_i \times a_i}\mathbb{C}$. The matrices g_i and h_i are inverse of each other, so they have to be square matrices, which shows $a_i = b_i$.□

Now let V be a continuous finite dimensional representation. One knows that $V^G \subset V$ as a subrepresentation, so $V = V^G \oplus W$, W is some G-representation. We would like to find out W explicitly. Define $\pi : V \to V$ by $\pi(v) = \int_G \rho(g) v dg$, or $\pi = \int_G \rho(g) dg$. π has following properties:

(1) $\forall v \in V$, $\pi(v) \in V^G$

Proof. $\rho(h)\pi(v) = \int_G \rho(h)\rho(g)vdg = \int_G \rho(gh)vdg = \int_G \rho(g)vdg = \pi(v)$. The third equality is due do the left invariant property of the Harr measure.

(2) If $v \in V^G$, then $\pi(v) = v$.

Proof. $\int_G \rho(g)vdg = \int_G vdg = v$. The last equality is due to the normality of Harr measure. This property says $\pi^2 = \pi$, and π is a section of $V^G \in V$.

(3) π commutes with G-action.

Proof. $\pi(\rho(h)v) = \int_G \rho(g)\rho(h)vdg = \int_G \rho(g)vdg = \pi(v) = \rho(h)\pi(v)$, where the last equality is due property (1).

So $\ker(\pi)$ is a G-subrepresentation and $V \cong V^G \oplus \ker(\pi)$. $\pi|_{V^G} = Id, \pi|_{\ker(\pi)} = 0$. This implies $Tr(\pi) = \dim V^G$. On the other hand, $Tr(\pi) = \int_G Tr(\rho(g))dg$. Therefore $\dim(V^G) = \int_G Tr(\rho(g))dg$.

Define the character of ρ to be the map $\chi : G \to \mathbb{C}$, $\chi(g) = Tr(\rho(g))$. The following are some examples and properties of characters.

Example 4. (1) *If* $V = \mathbb{C}$, $\rho(g) = Id$, *then* $\chi(g) = 1$

(2) *If* $V = \mathbb{C}^n$, $G = U(n)$, *then* $\chi(g) = Tr(g) = \sum_{j=1}^n e^{i\theta_j}$, *where* $e^{i\theta_j}$ *'s are the eigenvalues of* G.

(3) $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$

(4) $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$

(5) *Let* V^* *be the dual space of* V, *then* $\chi_{V^*}(g) = \overline{\chi_V(g)}$

(5) is not true for general topological groups. For a counterexample, let $G = \mathbb{C}^*$, $\rho(a) = (a)$ as a 1 by 1 matrix acting on V. Then $\chi(a) = a$. Then the action of a on V^* is multiplication by (a^{-1}). So $\chi_{V^*}(a) = a^{-1}$. On a compact subgroup: circle $\mathbb{S}^1 = \{e^{i\theta}\}$, $(e^{i\theta})^{-1} = e^{-i\theta}$. But on $\mathbb{C}^*$, $a^{-1} \neq \overline{a}$ for most a. Another example is $GL(2)$ acting on $V = \mathbb{C}^2$ by multiplication. $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ acts on V^* as multiplication of $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{smallmatrix}\right)$. Hence $\chi_V(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)) = a + b$, whereas $\chi_{V^*}(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)) = a^{-1} + b^{-1}$, which is another counterexample of (5). However, what is true for general group is $\chi_V(g) = \chi_{V^*}(g^{-1})$

Theorem 5. $\dim V^G = \int_G \chi_V(g)dg$.

One application of this theorem: Let V, W be G-representation acting on $Hom(U, W) \cong V^* \otimes W$, by $(g \cdot \phi)(v) = \rho_W(g) \cdot \phi \cdot \rho_V(g^{-1}) \cdot (v)$. Note $Hom(V, W)^G = Hom_G(V, W)$. So $\dim Hom_G(V, W) = \int_G \overline{\chi_V(g)}\chi_W(g)dg$.

Corollary 6. *If* V *and* W *are simple, then* $\int_G \overline{\chi_V(g)}\chi_W g dg = \begin{cases} 1, & \text{if } V \cong W \\ 0, & \text{if } V \ncong W \end{cases}$.

Corollary 7. *Characters of simple representations are orthonormal in* $\mathbb{C}^G$*, with Hermitian product* $\langle \phi, \varphi \rangle = \int_G \overline{\varphi(g)} \phi(g) dg$.

Corollary 8. *Characters of simple represntations are linearly independent in* $\mathbb{C}^G$*, where* $\mathbb{C}^G$ *is the set of functions from* G *to* $\mathbb{C}$.

Corollary 9. *If* $\chi_V = \chi_W$*, then* $V \cong W$.

Proof. Let $V = \oplus_i U_i^{\oplus a_i}$, $W = \oplus_i U_i^{\oplus b_i}$. Then $\sum_i a_i \chi_{U_i} = \sum_i b_i \chi_{U_i}$, by the last corollary, $a_i = b_i$. □