NOTES FOR OCTOBER 12

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The goal for today's lecture is to prove:

Theorem 1. *The characters of polynomial GL_n irreps are the Schur functions.*

The key will be to prove the following Peter-Weyl-like theorem

Theorem 2. *Consider the polynomial ring in n^2 variables z_{ij}. As a $GL_n \times GL_n$ representation, we have*

$$\mathbb{C}[z_{ij}] \cong \bigoplus_{V \textit{ a polynomial irrep}} V^\vee \otimes V.$$

As in Peter-Weyl, this sum means to take each isomorphism class once.
We continue the abbreviations

$$G = GL_n \quad K = U(n) \quad T = \{\mathrm{diag}(z_1, \ldots, z_n) : z_i \in \mathbb{C}^*\} \quad S = K \cap T = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})\}.$$

1. Proof of Theorem 2

We have a map $\mathbb{C}[z_{ij}] \to C^0(K)$ by restricting functions to the unitary group. Since polynomials in the z_{ij} are analytic functions, this map is injective by the key lemma from last time. We claim that it lands in $\mathcal{O}(K)$. Proof: $\mathbb{C}[z_{ij}] = \bigoplus_d \mathbb{C}[z_{ij}]_d$, where $\mathbb{C}[z_{ij}]_d$ is homogenous polynomials of degree d. Now, $\mathbb{C}[z_{ij}]_d$ is clearly a finite dimensional $K \times K$ subrep of $C^0(K)$. So, by results from October 8, it is in $\mathcal{O}(K)$.
Therefore, $\mathbb{C}[z_{ij}] \cong \bigoplus_{V \in S} V^\vee \otimes V$ for some set S of simple representations of K. We now must determine what the set S is.
Let V occur in $\mathbb{C}[z_{ij}]$. Looking at the $1 \times G$ action on V, it is clear that V is a polynomial G rep. So every representation $V \in S$ is the restriction of a polynomial representation of G.
On the other hand, if V is a polynomial representation of G, then the embedding $\mathrm{End}(V)^\vee \to C^0(G)$ clearly lands in $\mathbb{C}[z_{ij}]$. Explicitly, we are saying that $\lambda(\rho_V(g))$ is a polynomial in the z's, given that the entries of $\rho_v(g)$ are such a polynomial; that is obvious.
So we conclude that S is the set of polynomial representations of G as desired.

2. A combinatorial consequence

Consider both sides of Theorem 2 as $T \times T$ representations. To be precise, we are going to be acting by $\mathrm{diag}(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \times \mathrm{diag}(y_1, y_2, \ldots, y_n)$. (The inverses in the first term are precisely there to cancel the inverses defining the action of $G \times G$ on $C^0(G)$.)
On the left hand side, z_{ij} transforms by $x_i y_j$. So the character of the left hand side is

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - x_i y_j}.$$

On the right hand side, $\mathrm{diag}(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \times \mathrm{diag}(y_1, y_2, \ldots, y_n)$ acts on $V^\vee \otimes V$ by

$$\chi_{V^\vee}(x_1^{-1}, \ldots, x_n^{-1}) \chi_V(y_1, \ldots, y_n) = \chi_V(x_1, \ldots, x_n) \chi_V(y_1, \ldots, y_n).$$

So we deduce

$$\prod_{1 \leq i,j \leq n} \frac{1}{1 - x_i y_j} = \sum_{V \text{ a polynomial irrep}} \chi_V(x_1, \ldots, x_n) \chi_V(y_1, \ldots, y_n).$$

3. Finishing the proof

We would like to deduce that the χ_V are the Schur functions. There are two ways to finish the proof from here, both slightly more awkward than I would like.

Method 1. From a homework problem, $\chi_V(x_1, \dots, x_n)$ is a homogenous polynomial. As we noted in the previous class, we already know that the number of polynomal irreps of degree d is equal to the number of partitions of d. By a lemma proved way back on September 12, this means that the χ_V are self dual. Also, χ_V is in Λ by the previous class. By another lemma from September 12, a self dual basis of Λ must be $\pm s_\lambda$. It is clear that χ_V has nonnegative coefficients, so the plus sign is correct. □

Method 2. We don't really need to know that the number of degree d polynomial irreps is $p(d)$. Indeed, if f_i is any family of symmetric polynomials with integer coefficients obeying $\prod 1/(1 - x_i y_j) = \sum f_i(x) f_i(y)$, then I claim that the list of f_i contains each $\pm s_\lambda$ exactly once, plus possibly some occurrences of the 0 function. Proof sketch: Let $f_i = \sum_\lambda a_{i\lambda} s_\lambda$. Comparing coefficients of $s_\lambda(x) s_\lambda(y)$, we see that $\sum_i a_{i\lambda}^2 = 1$. So, for fixed λ, exactly one $a_{i\lambda}$ is ± 1 and the rest are zero. Comparing coefficients of $s_\lambda(x) s_\mu(y)$, we see that, for fixed i, at most one $a_{i,\lambda}$ is nonzero. So the χ_V are $\pm$ the s_λ, and maybe some zero functions. But it is clear that the χ_V are nonzero and have nonnegative coefficients, so again we win. □

4. Concluding comments

• If we look at the coordinate ring of GL_n, namely $\mathbb{C}[z_{ij}][\det^{-1}]$, we get $\bigoplus V^\vee \otimes V$ where the sum is over rational representations.
• The characters of the rational irreps are of the form

$$(x_1 x_2 \dots x_n)^{-N} s_\lambda(x_1, \dots, x_n).$$

Proof: Just tensor with a high power of the determinant representation to make it into a polynomial representation. We have

$$s_{(\lambda_1+1, \lambda_2+1, \dots, \lambda_n+1)}(x_1, x_2, \dots, x_n) = (x_1 x_2 \dots x_n) s_{\lambda_1, \lambda_2, \dots, \lambda_n}(x_1, x_2, \dots, x_n).$$

As a result, the same symmetric Laurent polynomial can be expressed using more than one pair (λ, N) as above. A nonredundant indexing set is the set of integer sequences $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$, where we do **not** impose that $\mu_n \geq 0$. The correspondence is that $\mu_i = \lambda_i - N$.
• It follows immediately from the above that $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$, since the Schurs are orthonormal.
• We can look at $\mathbb{C}[z_{ij}]$ where $1 \leq i \leq m$ and $1 \leq j \leq n$ as a $GL_m \times GL_n$ rep. We have the equality of generating functions

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x_1, \dots, x_m) s_\lambda(y_1, \dots, y_n).$$

(Just take the identity in infinitely many variables and stick in 0 for the appropriate x and y variables.) So

$$\mathbb{C}[z_{ij}] \cong \bigoplus_\lambda V_\lambda(m) \otimes V_\lambda(n)$$

where $V_\lambda(m)$ is the representation of GL_m with character $s_\lambda(x_1, \dots, x_m)$. The summands with $\ell(\lambda) > \min(m, n)$ are zero, so we can equivalently write

$$\mathbb{C}[z_{ij}] \cong \bigoplus_{\ell(\lambda) < \min(m,n)} V_\lambda(m) \otimes V_\lambda(n).$$