NOTES FOR OCTOBER 17, 2012

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We spent some time discussing Problem 5d on Problem Set 5. The analogous finite group statement is the following:

> *Let V be a faithful G-representation. Set $W = V \oplus V^\vee \oplus \mathbb{C}$. Then every irreducible representation U occurs in $W^{\oplus N}$ for $N \gg 0$.*

1. Some comments from last time

$$\mathbb{C}[x_{ij}] = \bigoplus V^\vee \otimes V,$$

where V varies over polynomial irreducible representations of GL_n and the equality above is taken as $(\mathrm{GL}_n \times \mathrm{GL}_n)$-representations. We want to point out that we similarly have:

$$\mathbb{C}[z_{ij}]_{1\le i\le m, 1\le j<n} = \bigoplus_{\lambda,\, \ell(\lambda)\le \min(m,n)} V_\lambda^\vee(m) \otimes V_\lambda(n)$$

as $(\mathrm{GL}_n \times \mathrm{GL}_n)$-representations.

Proof. We have

$$\prod_{i=1}^m \prod_{j=1}^n \frac{1}{1 - x_i y_j} = \sum_\lambda S_\lambda(x_1, \ldots, x_m) S_\lambda(y_1, \ldots, y_n).$$

Note that $S_\lambda(x_1, \ldots, x_m) = 0$ if $\ell(\lambda) > m$ and $S_\lambda(y_1, \ldots, y_n)$ if $\ell(\lambda) > n$. The result follows. □

Last time that we showed that we have a bijective correspondence between polynomial GL_n representations and partitions λ with $\ell(\lambda) \le n$; and the character of V_λ is s_λ. We want to point out that this implies:

$$\dim \mathrm{Hom}_{\mathrm{GL}}(V, W) = \langle \chi_V, \chi_W \rangle.$$

(Consider χ_V and χ_W as polynomials in Λ using the combinatorially defined inner product on Λ.) Note that if $|\lambda| > n$, the we use the standard inclusion $\Lambda_n \hookrightarrow \Lambda$.

Notation. V_λ or $V_\lambda(n)$ is the GL_n-representation with character $S_\lambda(x_1, \ldots, x_n)$.

2. How we will construct V_λ

Recall that

$$h_\lambda = s_\lambda + \sum_{\mu \prec \lambda} \kappa_{\lambda\mu} s_\mu,$$

$$e_{\lambda^{\mathrm{T}}} = s_\lambda + \sum_{\mu \succ \lambda} \kappa_{\lambda^{\mathrm{T}}\mu^{\mathrm{T}}} s_\mu.$$

So the equality

$$\langle h_\lambda, e_{\lambda^{\mathrm{T}}} \rangle = 1$$

comes from the s_λ term.

Let

$$H = \bigotimes\nolimits_k \mathrm{Sym}^{\lambda_k} V,$$

$$E = \bigotimes\nolimits_k \Lambda^{\lambda_k^{\mathrm{T}}} V.$$

So $\chi_H = h_\lambda$ and $\chi_E = e_{\lambda^T}$. We see that

$$H = V_\lambda \oplus \bigoplus_{\mu \prec \lambda} V_\mu^{\oplus \kappa_{\lambda\mu}},$$

$$E = V_\lambda \oplus \bigoplus_{\mu \succ \lambda} V_\mu^{\oplus \kappa_{\lambda^T \mu^T}}.$$

So $\mathrm{Hom}_{\mathrm{GL}(V)}(E, H) \cong \mathbb{C}$ and if φ is a $\mathrm{GL}(V)$-equivariant homomorphism $E \to H$, then $\mathrm{Im}(\varphi) \cong V_\lambda$. Our next goal will be to describe such a map φ explicitly.