NOTES FOR OCTOBER 19, 2012: SCHUR FUNCTORS

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1. Homework questions

Is Problem 1 as simple as it appears?

Answer: Yes. $g(x_1,\dots,x_k,y_1,\dots,y_{n-k}) = f(x_1,\dots,x_k,y_1,\dots,y_{n-k})$.

Can something be said about expanding $s_\nu(x_1,\dots,x_k,y_1,\dots,y_{n-k})$ in the basis of products $s_\lambda(x_1,\dots,x_k)s_\mu(y_1,\dots,y_{n-k})$? Answer: Yes! These are the Littlewood-Richardson coefficients.

$$s_\nu(x_1,\dots x_k,y_1,\dots,y_{n-k}) = \sum_{\lambda,\mu} c^\nu_{\lambda\mu} s_\lambda(x)s_\mu(y)$$

They are all the coefficients of multiplication:

$$s_\lambda(z)s_\mu(z) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(z).$$

This is not obvious, but you know enough to prove it.

Is it always true that representations of $G\times H$ can be broken up into $\bigoplus U_i\otimes V_i$, where U_i and V_i are representations of G and H respectively?

Answer: If for groups G and H, every representation is a direct sum of simples, then $G\times H$ also has this property, and $G\times H$ simples are of the form $V\otimes W$, where V is G-simple and W is H-simple.

Proof sketch: Let X be $G\times H$ simple. Let $V\subset X$ be a G-simple subrepresentation. Define $W = \mathrm{Hom}_G(V,X)$. Map $V\otimes W\to X$ as a $G\times H$ representation. The map is surjective since X simple. It is injective by looking at $X = V\oplus\cdots\oplus V\oplus U$. $\square$

Argument stolen from `http://math.stackexchange.com/questions/136048`. This seems to be the sort of thing everyone knows, but doesn't appear in enough texts.

Some additional points not mentioned in class: If we are working over a field which is not algebracally closed, then simple $\otimes$ simple need not be simple. EG, let $G=\mathbb{Z}/3$ acting on a two dimensional real vector space V by $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$. Then V is simple (over $\mathbb{R}$) but $V\otimes_\mathbb{R} V$ is not simple. However, over $\mathbb{C}$, it is true that simple $\otimes$ simple is simple. For finite groups, this is an easy application of character theory; it is actually true for all groups.

For groups whose representations are not all direct sums of simples, not every rep breaks up as a direct sum of tensor products. For example, let G be the additive group $\mathbb{Z}$ and let $G\times G$ act on $\mathbb{C}^2$ by $\rho(j,k) = \begin{pmatrix} 1 & j+k \\ 0 & 1\end{pmatrix}$. This representation cannot be written as a nontrivial direct sum; and it is not the tensor product of a G-rep with another G-rep.

2. Finding the irrep V_λ (from last class)

Let λ be a partition of n. We want to construct V_λ, the $GL(n)$ irrep with character s_λ. Let $N=|\lambda|$, and let $V=\mathbb{C}^n$. Define the two $GL(n)$-representations

$$H = \bigotimes_k \mathrm{Sym}^{\lambda_k} V \qquad \text{and} \qquad E = \bigotimes_k \bigwedge^{(\lambda^T)_k} V$$

which have characters $\chi_H = h_\lambda$ and $\chi_E = e_{\lambda^T}$, respectively. Recall that

$$h_\lambda = s_\lambda + \sum_{\mu\prec\lambda} \kappa_{\lambda\mu} s_\mu \qquad \text{and} \qquad e_{\lambda^T} = s_\lambda + \sum_{\mu\succ\lambda} \kappa_{\lambda^T\mu^T} s_\mu$$

so the equality $\langle h_\lambda, e_{\lambda^T}\rangle$ comes from the s_λ term. It follows that the only $GL(n)$ irrep that H and E have in common is a single copy of V_λ. Any non-zero $GL(n)$-equivariant map $E \to H$ or $H \to E$ is actually a map from one copy of V_λ to the other copy of V_λ. Our goal is therefore to construct such a map.

3. A $GL(n)$-EQUIVARIANT MAP $E \to H$

We will construct a non-zero $GL(V)$-equivariant map $E \to H$. The image of this map will be the copy of V_λ in H.

Let's think of $\operatorname{Sym}^k V$ and $\bigwedge^k V$ concretely. Note that $\operatorname{Sym}^k V$ is can be thought of as either a subspace or a quotient of $V^{\otimes k}$. Viewing $\operatorname{Sym}^k V$ as the subspace of $V^{\otimes k}$ of S_k-invariant tensors, there is the inclusion

$$\operatorname{Sym}^k V \to V^{\otimes k}$$
$$v_1 \cdots v_k \mapsto \frac{1}{k!}\sum_{w \in S_k} v_{w(1)} \otimes \cdots \otimes v_{w(k)}.$$

(Notation: We're writing elements of $\operatorname{Sym}^k V$ as monomials. This is not standardized.) Viewing $\operatorname{Sym}^k V$ as a quotient of $V^{\otimes n}$, we have the projection map

$$V^{\otimes k} \to \operatorname{Sym}^k V$$
$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \cdots v_k$$

which equates different permutations of a tensor. Similarly for $\bigwedge^k V$, there are maps $\bigwedge^k V \to V^{\otimes n}$ and $V^{\otimes n} \to \bigwedge^k V$ defined by

$$v_1 \wedge \cdots \wedge v_k \mapsto \frac{1}{k!}\sum_{w \in S_k} (-1)^w v_{w(1)} \otimes \cdots \otimes v_{w(k)}$$
$$v_1 \otimes \cdots \otimes v_k \mapsto v_1 \wedge \cdots \wedge v_k$$

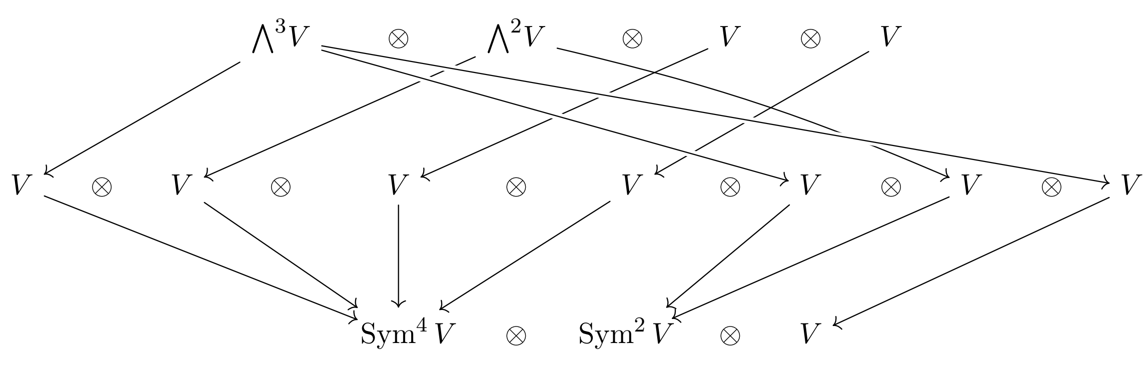
so that $\bigwedge^k V$ can also be viewed as either a subspace or a quotient of $V^{\otimes n}$. (Note: $(-1)^w$ is the parity of the permutation.)

The map $E \to H$ is constructed out of the two parts $E \to V^{\otimes N} \to H$, inclusion and projection. The cells of a Young tableau of shape λ index the components of $V^{\otimes N}$ (recall that $N = |\lambda|$). The columns index the components of E, and the rows index the components of H. For the map $E \to V^{\otimes N} \to H$, include from E into $V^{\otimes N}$ by column, and project from $V^{\otimes N}$ to H by row.

We give this construction by example. Consider the following partition:

$$\lambda = (4,2,1) \qquad \lambda^T = (3,2,1,1) \qquad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

The leftmost column of the tableau corresponds with $\bigwedge^3 V$, the first component of E. It maps to the first, fifth, and seventh components of $V^{\otimes N}$, which in turn project to $\operatorname{Sym}^4 V$, $\operatorname{Sym}^2 V$, and V, respectively (the first, second, and third rows). Or we can look at the following picture:



We need this 'twisting' to get non-zero map.

For a smaller example, consider

$$\lambda = (2,1) \qquad \lambda^T = (2,1) \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and the picture

$$\begin{tikzcd}
 & \bigwedge^2 V \arrow[dl] \arrow[drrr] & \otimes & V \arrow[dl] & \\
V \arrow[dr] & \otimes & V \arrow[dl] & \otimes & V \arrow[dl] \\
 & \mathrm{Sym}^2 V & \otimes & V &
\end{tikzcd}$$

which is simple enough that we will write the map explicitly. The component of the map from $\bigwedge^2 V$ to $V \otimes V$ is $u \wedge v \mapsto \frac{1}{2}(u \otimes v - v \otimes u)$. On an arbitrary pure tensor in $\bigwedge^2 V \otimes V$, the whole map is

$$\begin{array}{ccccc} \bigwedge^2 V \otimes V & \to & V \otimes V \otimes V & \to & \mathrm{Sym}^2 V \otimes V \\ (u \wedge v) \otimes w & \mapsto & \frac{1}{2}(u \otimes w \otimes v - v \otimes w \otimes u) & \mapsto & \frac{1}{2}((uw) \otimes v - (vw) \otimes u). \end{array}$$

One special case is

$$(u \wedge v) \otimes w + (v \wedge w) \otimes u + (w \wedge u) \otimes v \mapsto 0$$

(which is suggestive of the Jacobi identity).

Since $E \to H$ is a $GL(V)$-equivariant map, it commutes with torus action. In weight $x_i^2 x_j$, E has one eigenvector $(e_i \wedge e_j) \otimes e_i$, and its image is non-zero. In weight $x_i x_j x_k$, E has 3 eigenvectors, $(e_i \wedge e_j) \otimes e_k$, $(e_j \wedge e_k) \otimes e_i$, $(e_k \wedge e_i) \otimes e_j$ and their images span a 2 dimensional subspace of H. The corresponding Schur function is

$$s_{21}(x) = \sum x_i^2 x_j + 2 \sum x_i x_j x_k.$$

4. Other approaches (and the Young symmetrizer)

- We could map $H \to E$ instead.
- We could try to map both E and H to $V^{\otimes N}$ and intersect their images. But this might not work. This is because even though both E and H have a copy of V_λ, their images in $V^{\otimes N}$ might be isomorphic, but not the same, in which case their images would not intersect.
- We could think of H and E as subspaces of $V^{\otimes N}$. Let a_λ be projection onto $H \subset V^{\otimes N}$, and b_λ be projection onto $E \subset V^{\otimes N}$. Look at the image of $a_\lambda b_\lambda$. (Note the image of $b_\lambda a_\lambda$ will be isomorphic to $a_\lambda b_\lambda$ but not equal, unless E did meet H. This is the same issue as in the previous bullet point.)

We'll do the third option. What is a_λ? It is

$$a_\lambda : V^{\otimes N} \to H \to V^{\otimes N},$$

the composition of projection and inclusion. It projects from $V^{\otimes N}$ to a copy of H inside $V^{\otimes N}$. Explicitly, the map is

$$a_\lambda(v_1 \otimes \cdots \otimes v_N) = \frac{1}{\lambda_1! \cdots \lambda_k!} \sum_{w \in S_{\lambda_1} \times \cdots \times S_{\lambda_k}} V_{w(1)} \otimes \cdots \otimes V_{w(N)}$$

with a sum over permutations $w \in S_N$ that preserve the rows of the λ-tableau. For example, with $\lambda = (2,1)$ the map is given by

$$a_{21} : v_1 \otimes v_2 \otimes v_3 \mapsto (v_1 v_2) \otimes v_3 \mapsto \tfrac{1}{2}(v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3).$$

Similarly, b_λ is

$$b_\lambda : V^{\otimes N} \to E \to V^{\otimes N}$$
$$b_\lambda(v_1 \otimes \cdots \otimes v_N) = \frac{1}{(\lambda^T)_1! \cdots (\lambda^T)_\ell!} \sum_w (-1)^w \, V_{w(1)} \otimes \cdots \otimes V_{w(N)}$$

with a sum over permutations $w \in S_N$ that preserve the columns of the λ-tableau, so that b_λ projects from $V^{\otimes N}$ to a copy of E inside $V^{\otimes N}$. For example,

$$b_{21} : v_1 \otimes v_2 \otimes v_3 \mapsto (v_1 \wedge v_3) \otimes v_2 \mapsto \tfrac{1}{2}(v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1)$$

where we can see same 'twisting' that we had in the earlier construction.

The composition $a_\lambda b_\lambda$ of both projections is called the ***Young symmetrizer***, and is written c_λ. We can think of V_λ as the image of c_λ in $V^{\otimes N}$.

5. FUNCTORIALITY

For any partition λ, we claim that

$$V \mapsto V_\lambda$$

is a functor (it is called the ***Schur Functor*** $\mathbb{S}_\lambda$).

We first want to show that

$$(c_\lambda)^2 = k_\lambda c_\lambda$$

for some constant k_λ, so that c_λ is almost an idempotent.

Think of V_λ as a subset of $V^{\otimes N}$. Write c_λ as a composition of projection and inclusion

$$c_\lambda : V^{\otimes N} \xrightarrow{\pi} V_\lambda \xrightarrow{i} V^{\otimes N}.$$

The map

$$V_\lambda \xrightarrow{i} V^{\otimes N} \xrightarrow{\pi} V_\lambda$$

is between irreducible representations, and by Schur's lemma

$$\pi \circ i = k_\lambda \operatorname{Id}$$

for some constant k_λ. Then

$$(c_\lambda)^2 = (i \circ \pi) \circ (i \circ \pi) = i \circ (\pi \circ i) \circ \pi = k_\lambda (i \circ \pi) = k_\lambda c_\lambda$$

as desired. (Because i is inclusion, the constant k_λ 'belongs' to π.)

This constant k_λ does not depend on V. Indeed, we can think about a_λ, b_λ, and c_λ as elements of $\mathbb{C}[S_n]$, e.g.

$$a_\lambda = \frac{1}{\lambda_1! \cdots \lambda_k!} \left(\sum_{w \in S_{\lambda_1} \times \cdots \times S_{\lambda_k}} w \right) \in \mathbb{C}[S_n]$$

so that the computation $(c_\lambda)^2 = (a_\lambda b_\lambda)^2$ is independent of the choice of V.

Now we show functoriality. Any linear map $\alpha : U \to V$ lifts to a map $U_\lambda \to V_\lambda$.

$$\begin{tikzcd} U_\lambda \arrow[r, "i"] & U^{\otimes N} \arrow[d, "\alpha^{\otimes N}"] \\ V_\lambda & V^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"'] \end{tikzcd}$$

Let $\beta : V \to W$ be another map, and consider the composition $U_\lambda \to V_\lambda \to W_\lambda$.

$$
\begin{tikzcd}
U_\lambda \arrow[r, "i"] & U^{\otimes N} \arrow[d, "\alpha^{\otimes N}"] \\
V_\lambda \arrow[dr, "i"'] & V^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"'] \\
 & V^{\otimes N} \arrow[d, "\beta^{\otimes N}"] \\
W_\lambda & W^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
U_\lambda \arrow[r, "i"] & U^{\otimes N} \arrow[d, "\alpha^{\otimes N}"] \\
 & V^{\otimes N} \arrow[d, "\frac{1}{k_\lambda}c_\lambda"] \\
 & V^{\otimes N} \arrow[d, "\beta^{\otimes N}"] \\
W_\lambda & W^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
U_\lambda \arrow[r, "i"] & U^{\otimes N} \arrow[d, "\alpha^{\otimes N}"] \\
 & V^{\otimes N} \arrow[d, "\beta^{\otimes N}"] \\
 & W^{\otimes N} \arrow[d, "\frac{1}{k_\lambda}c_\lambda"] \\
W_\lambda & W^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"]
\end{tikzcd}
$$

We've made 'c_λ commute with β'. Also $\frac{1}{k_\lambda}\pi \circ \frac{1}{k_\lambda}c_\lambda = \frac{1}{k_\lambda}\pi$, so

$$
\Longrightarrow \quad
\begin{tikzcd}
U_\lambda \arrow[r, "i"] & U^{\otimes N} \arrow[d, "\alpha^{\otimes N}"] \\
 & V^{\otimes N} \arrow[d, "\beta^{\otimes N}"] \\
W_\lambda & W^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"]
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
U_\lambda \arrow[r, "i"] & U^{\otimes N} \arrow[d, "(\beta \circ \alpha)^{\otimes N}"] \\
W_\lambda & W^{\otimes N} \arrow[l, "\frac{1}{k_\lambda}\pi"]
\end{tikzcd}
$$

and we have functoriality.