LECTURE 20: REALIZING SCHUR FUNCTOR USING MATRIX MINORS

In this lecture, $G = GL(V)$, $|\lambda| = N$, $V = \mathbb{C}^n$, $E = \otimes_k(\wedge^{(\lambda^T)_k} V)$, $H = \otimes_k(Sym^{\lambda_k} V)$. We would like to find a $GL(v)$ equivariant homomorphism from E to H. The strategy is to work inside $\mathbb{C}[z_{ij}]$, where $1 \leq i, j \leq n$, and build a $1 \times G$ subrepresentation which is isomorphic to V_λ.

Embed H into $\mathbb{C}[z_{ij}]$ as polynomials which have deg λ_i in $z_{i1}, z_{i2}, \ldots, z_{in}$ for $1 \leq i \leq l(\lambda)$. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then send $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{l(\lambda)}}$ to the determinant of the $l \times l$ submatrix using top l rows and column i_j for $1 \leq j \leq l = l(\lambda)$, and extend by linearity to get a map from E to $\mathbb{C}[z_{ij}]$. Note that the image of E in $\mathbb{C}[z_{ij}]$ lands in the image of H in $\mathbb{C}[z_{ij}]$.

Example 1. $\lambda = \ydiagram{4,2,1}$, V_{421} is spanned by all products of the form

$$\begin{vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{2i} & z_{2j} & z_{2k} \\ z_{3i} & z_{3j} & z_{3k} \end{vmatrix} \cdot \begin{vmatrix} z_{1l} & z_{1m} \\ z_{2l} & z_{2m} \end{vmatrix} \cdot z_{1p} \cdot z_{1q}$$

One can see this product has deg 4 in $z_{11}, \ldots z_{1n}$, deg 2 in $z_{21}, \ldots, z_{2n}$, deg 1 in $z_{31}, \ldots, z_{3n}$, thus lie in the image of H.

Note that it is now easy to see that we have a nonzero map $E \to H$: These products of minors are clearly nonzero (as long as $\ell(\lambda) \leq n$, so we can form large enough determinants inside an $n \times n$ matrix).

Example 2. $\lambda = \ydiagram{2,1}$. V_{21} is spanned by

$$\begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{1i}, \begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{1k}$$

where i, j, k are distinct, with the relation

$$\begin{vmatrix} z_{1i} & z_{1j} \\ z_{2i} & z_{2j} \end{vmatrix} \cdot z_{1k} - \begin{vmatrix} z_{1i} & z_{1k} \\ z_{2i} & z_{2k} \end{vmatrix} \cdot z_{1j} + \begin{vmatrix} z_{1j} & z_{1k} \\ z_{2j} & z_{2k} \end{vmatrix} \cdot z_{1i} = 0$$

The last equation exactly corresponds to the fact $(e_i \wedge e_j) \otimes e_k + (e_j \wedge e_k) \otimes e_i + (e_k \wedge e_i) \otimes e_l$ will be mapped to 0 under the that from E to H from last class. And it is easier to see now because this equation is the expansion of $\begin{vmatrix} z_{1i} & z_{1j} & z_{1k} \\ z_{1i} & z_{1j} & z_{1k} \\ z_{2i} & z_{2j} & z_{2k} \end{vmatrix}$ which is 0.

Example 3. $\lambda = \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}$, a column of n boxes, then one realizes V_{1^k} as span of $k \times k$ top justified minors. This is isomorphic to $\wedge^k V$.

Example 4. $\lambda = \overbrace{\begin{array}{|c|c|c|}\hline & \cdots & \\ \hline\end{array}}^{k}$, then V_k is realized as polynomials of degree k in $z_{11}, \ldots, z_{1n}$.

Example 5. If $\lambda = \overbrace{(k, k, \ldots, k)}^{d}$, the image V_λ in $\mathbb{C}[z_{ij}]$ is spanned by the k-fold products of the $d \times d$ top justified minors.

Since the image of V_λ will not use any z_{ij} for $i > d$, $V_\lambda \subset \mathbb{C}[z_{ij}]_{1 \leq i \leq d, 1 \leq j \leq n}$. Note that $\mathbb{C}[z_{ij}]_{1 \leq i \leq d, 1 \leq j \leq n} \cong \oplus_\mu V_\mu(d)^\vee \otimes V_\mu(n)$ as a $GL(d) \times GL(n)$ representation. $V_\lambda(d)$ can be realized as $(\det)^{\otimes k}$. Hence, one can identify $V_\lambda(d)^\vee \otimes V_\lambda(n) = \{f \in \mathbb{C}[z_{ij}]_{1 \leq i \leq d, 1 \leq j \leq n} | f(gM) = (\det g)^k f(M)\}$, where $M = (z_{ij})_{1 \leq i \leq d, 1 \leq j \leq n}$.

Let's think about the ring of invariants for the SL_d action on $\mathbb{C}[z_{ij}]_{1 \leq i \leq d, 1 \leq j \leq n}$. Note that we have

$$\left\{ f \in \mathbb{C}[z_{ij}]_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}} \right\}^{SL_d} = \bigoplus_k \left\{ f \in \mathbb{C}[z_{ij}]_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}} | f(gM) = (\det g)^k f(M) \right\}$$

Therefore, $\{f \in \mathbb{C}[z_{ij}]_{1 \leq i \leq d, 1 \leq j \leq n}\}^{SL_d}$ is spanned by the $GL(d) \times GL(n)$ orbit of k-fold product of $d \times d$ minors, where k runs over all nonnegative integers. In other words, the ring of SL_d invariants is generated by the $d \times d$ minors.

Can we understand what $V_\lambda^\vee \otimes V_\lambda$ looks like inside $\mathbb{C}[z_{ij}]$? In a not very useful sense, yes. We have constructed a vector in $V_\lambda^\vee \otimes V_\lambda$, so we know that $V_\lambda^\vee \otimes V_\lambda$ is the span of the $G \otimes G$ orbit of

$$\prod_{i=1}^{l(\lambda^T)} \begin{vmatrix} z_{11} & \cdots & z_{1(\lambda^T)_i} \\ \vdots & \ddots & \vdots \\ z_{(\lambda^T)_i 1} & \cdots & z_{(\lambda^T)_i (\lambda^T)_i} \end{vmatrix}$$

This is usually not useful, but there are a few cases where we can unpack it.

Example 6. If $\lambda = (1^k)$, $V_\lambda^\vee \otimes V_\lambda$ has a basis of $k \times k$ minors $\begin{vmatrix} z_{i_1 j_1} & \cdots & z_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ z_{i_k j_1} & \cdots & z_{i_k j_k} \end{vmatrix}$, where $i_1 < i_2 < \ldots < i_k, j_1 < j_2 < \ldots < j_k$.

Example 7. If $\lambda = k$, $V_\lambda^\vee \otimes V_\lambda$ has a basis $k \times k$ perminants *per* $\begin{pmatrix} z_{i_1 j_1} & \cdots & z_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ z_{i_k j_1} & \cdots & z_{i_k j_k} \end{pmatrix}$, where $i_1 \leq i_2 \ldots \leq i_k, j_1 \leq j_2 \leq \ldots \leq j_k$.

In general, how can we think about our construction of V_λ in terms of Peter-Weyl? We have embedded V_λ into $\mathbb{C}[z_{ij}]$ as a $1 \times G$ subrepresentation. It must be some $u \otimes V_\lambda$ for some $u \in V_\lambda$. Let N_- be the set of lower triangular matrix with 1's on the diagonal. Notice if $f \in V_\lambda$, $g \in N_-$, then $f(g^{-1}M) = f(M)$. So u is N_- invariant. For W a $GL(n)$ representation, a vector in W in which is N_--invariant is called a *low weight vector*.

Theorem 8. *If W is a $GL(n)$ irreducible representation, then* $\dim W^{N_-} = 1$.

Preview for Wednesday: Given a semistandard Young tableau, under the map we constructed today, one can map it to a product of minors. For example, if $\lambda = (4, 2, 1)$, and the SSYT is $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 5 & 7 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$, polynomial that this SSYT corresponds with is $\begin{vmatrix} z_{11} & z_{12} & z_{14} \\ z_{21} & z_{22} & z_{24} \\ z_{31} & z_{32} & z_{34} \end{vmatrix} \cdot \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} \cdot z_{15} \cdot z_{17}$. The claim on Wednesday is that these polynomials form a basis of V_λ.