NOTES FOR OCTOBER 24

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1. Setup

- Let λ be a partition with $l(\lambda) \leq n$. We embed $V_\lambda(n)$ into $\mathbb{C}[z_{ij}]_{1\leq i,j\leq n}$ as the span of all products $\det|\text{stuff}|\det|\text{stuff}|\cdots\det|\text{stuff}|$ where the kth factor is a $(\lambda^\top)_k \times (\lambda^\top)_k$ top justified submatrix of $\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}$.

- We use $\Delta_{i_1\cdots i_k}$ to denote $\det \begin{pmatrix} z_{1i_1} & \cdots & z_{1i_k} \\ \vdots & \ddots & \vdots \\ z_{ki_1} & \cdots & z_{ki_k} \end{pmatrix}$.

- For T a tableaux of shape λ, let Δ_T denote the product $\Delta_{\_}\cdots\Delta_{\_}$ where the subscript for the kth factor is the kth column of T.

  Example: $\Delta_{\begin{smallmatrix}1&1&2&5\\2&4&&\\3&&&\end{smallmatrix}} = \Delta_{123}\Delta_{14}\Delta_2\Delta_5$

2. Main Theorem

Our goal for this lecture is to prove: **Theorem:** As T ranges over SSYT of shape λ with entries from 1 to n, the Δ_T form a basis for $V_\lambda(n)$.

Our first step in proving this theorem will be to put an order on the monomials in $\mathbb{C}[z_{ij}]$. Represent each monomial in the z_{ij} as an $n\times n$ matrix whose (i,j)th entry is the exponent of z_{ij} in the monomial. Order the monomials using the the lexicographical order reading left to right then top to bottom on these matrices.

Example: $z_{11}z_{23}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} > \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = z_{12}z_{21}^2$

We use this ordering since it has 2 key properties:
Key Properties:

- For any monomial M and any $i<j$ and $k<l$, we have $Mz_{ik}z_{jl} > Mz_{il}z_{jk}$.

- $M \geq M'$ and $N \geq N'$ then $MN \geq M'N'$.
  Note: This implies that leading term(fg) = leading term(f) $\times$ leading term(g)

Example:

$$\Delta_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \Delta_{13}\Delta_2 = (z_{11}z_{23} - z_{13}z_{21})z_{12}$$
$$= z_{11}z_{23}z_{12} - z_{13}z_{21}z_{12}$$

We have

$$\begin{pmatrix} 1 & 1 & \\ & & 1 \end{pmatrix} > \begin{pmatrix} & 1 & 1 \\ 1 & & \end{pmatrix}$$

so the leading monomial for $\Delta_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}$ is $z_{11}z_{23}z_{12}$.

We will first prove that the Δ_T for $T \in \mathrm{SSYT}(\lambda)$ are linearly independent in $V_\lambda(n)$ and then show that they span $V_\lambda(n)$.

3. Linear Independence

Claim: If T and U are distinct SSYT of shape λ then leading term $(\Delta_U) \neq$ leading term (Δ_T).

Proof: We will prove this by showing that each monomial can be the leading term of at most one Δ_T. Let e be an $n \times n$ matrix with entries in $\mathbb{Z}_{\geq 0}$ representing the leading monomial for a SSYT T. We will reconstruct T column by column from e. The entries in the first column of T must correspond to the nonzero exponents in the first $(\lambda^\top)_1$ rows of e. T is a SSYT so the entry in row r and column 1 of T must correspond to the left most nonzero exponent in row r of e. Use this to reconstruct the first column of T. Subtract the matrix corresponding the first column of T from e and repeat until you're left with the zero matrix.

The linear independence of the Δ_T over $T \in \mathrm{SSYT}(\lambda)$ follows from this claim. Write down a matrix whose columns correspond to the monomials in z_{ij}'s ordered by $<$ and whose rows correspond to Δ_T's for $T \in \mathrm{SSYT}(\lambda)$ ordered by $<$ on their leading terms. The claim tells us that the submatrix of columns which are leading terms is an upper triangular matrix with 1's on the diagonal. This matrix therefore has full rank, so the Δ_T must be linearly independent.

4. Spanning

We still need to show that the Δ_T, $T \in \mathrm{SSYT}(\lambda)$, span $V_\lambda(n)$. (Note: This follows from the fact that the Δ_T are linearly independent and there are $\dim V_\lambda(n)$ of them, but we will provide a more constructive and useful proof).

We already know that Δ_T, T any tableaux of λ, span $V_\lambda(n)$. So we just need to construct a method to express Δ_T when T is not a SSYT as a linear combination of Δ_U, $U \in \mathrm{SSYT}(\lambda)$.

Claim: Let $s \geq t > 0$. Let $I = I_1 \sqcup I_2$ and $J = J_1 \sqcup J_2$ with $|I| = s$, $|J| = t$, and $|I_2| + |J_1| = s+1$. Let $I_2 \cup J_1 = \{r_1 r_2 \cdots r_{s+2}\}$ where elements can be repeated. Then

$$\sum_{\omega \in S_{s+1}} (-1)^\omega \Delta_{I_1 r_{\omega(1)} r_{\omega(2)} \cdots r_{\omega(|I_2|)}} \Delta_{r_{\omega(|I_2|+1)} \cdots r_{\omega(s+1)} J_2} = 0$$

Proof: This expression is an antisymmetric multilinear function of the $s+1$ vectors of the columns of $\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}$ indexed by $I_2 \cup J_1$, and only it uses their top s entries. So it is an element of $\wedge^{s+1}$ of a dimension s vector space, so it is 0.

Example:

$I = \{1\} \sqcup \{2,3\} = I_1 \sqcup I_2$, $s = 3$, and $J = \{4,5\} \sqcup \emptyset = J_1 \sqcup J_2$, $t = 2$.

Note: The terms in the sum are constant on the cosets of $S_{|I_2|} \times S_{s+1-|I_2|}$, so we can restrict to just summing over these cosets.

$$\begin{matrix} \Delta_{123}\Delta_{45} & -\Delta_{124}\Delta_{35} & +\Delta_{125}\Delta_{34} \\ +\Delta_{134}\Delta_{25} & -\Delta_{135}\Delta_{24} & +\Delta_{145}\Delta_{23} \end{matrix} = 0$$

$$\Delta_{\begin{smallmatrix}1&4\\2&5\\3&\end{smallmatrix}} - \Delta_{\begin{smallmatrix}1&3\\2&5\\4&\end{smallmatrix}} + \Delta_{\begin{smallmatrix}1&3\\2&4\\5&\end{smallmatrix}} + \Delta_{\begin{smallmatrix}1&2\\3&5\\4&\end{smallmatrix}} - \Delta_{\begin{smallmatrix}1&2\\3&4\\5&\end{smallmatrix}} + \Delta_{\begin{smallmatrix}1&2\\4&3\\5&\end{smallmatrix}} = 0$$

We can use this relation to express a non SSYT, $\Delta_{\begin{smallmatrix}1&2\\4&3\\5&\end{smallmatrix}}$, by a linear combination of SSYT terms.

Next lecture we will use this relation to produce an algorithm that will take Δ_T for T any non-semistandard tableaux, and express it as a sum of terms for SSYT, thus completing the proof that Δ_T for $T \in \mathrm{SSYT}(\lambda)$ span $V_\lambda(n)$.