NOTES FOR OCTOBER 26

RACHEL KARPMAN

1. The Semistandard Basis

Last time: we had a fixed partition λ, and T a filling of λ by integers $1, \dots, n$. For example, we might use

1	3	2	4
2	1		
3			

We constructed a polynomial Δ_T. For this example, we get

$$\Delta_T = \Delta_{125}\Delta_{31}\Delta_2\Delta_4$$

where

$$\Delta_{i_1,\dots,i_k} = \det \begin{pmatrix} z_{1i_1} & \cdots & z_{i_k} \\ \vdots & \ddots & \vdots \\ z_{ki_1} & \cdots & z_{ki_n} \end{pmatrix}$$

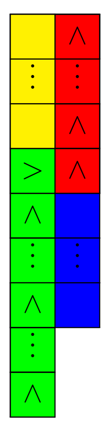
We were showing that $\{\Delta_T\}_{T\text{semistandard}}$ form a basis.

For this, we order the tableaux lexicographically, reading down the columns in order from left to right. That is, we read off the entries in the filling T in the order shown below

1	4	6	7
2	5		
3			

We will show that if T is not semistandard, then $\Delta_T \in \mathrm{Span}_{U<T}(\Delta_U)$.

If any column of T is not increasing, then sorting it products a small T' and $\Delta_T = \pm\Delta_{T'}$. So we may assume that the columns are increasing. If T is not semistandard, then we have two adjacent columns like this:



Break these columns up into $I_1 \sqcup I_2$, $J_1 \sqcup J_2$, where I_1 is yellow in the diagram above; I_2 is green; J_1 is red; and J_2 is blue.

Last time, we saw that

$$\Delta_{I_1 \sqcup I_2}\Delta_{J_1 \sqcup J_2} = \sum \pm \Delta_{I_1,()}\Delta_{(),J_2}$$

These equations are called "Plücker relations."

Multiplying by the polynomials corresponding to the columns that remain unchanged, the conclusion follows.

2. Preview: Invariants

(David writes: This section was pretty confused. See the notes I posted on the webpage for a clearer presentation. Hopefully, Kevin will clear a lot of this up while I'm away.)

We saw on homework that for $\dim V \gg n$, $\mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n}) \cong \mathbb{C}[S_n]$. We will eventually show this is true for all values of $\dim V$.

Recall: if $W \circlearrowleft G$, then we have an action of G on $\mathrm{Hom}(W, \mathbb{C})$ by $(g \cdot \varphi)(w) = \varphi(g^{-1}w)$.

Now, $\mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n}) \cong \mathbb{C}[S_n]$ is naturally isomorphic to $\mathrm{Hom}_{GL(V)}(V^{\otimes n} \otimes (V^{\vee})^{\otimes n}, \mathbb{C})$, which has a basis

$$(u_1 \otimes \cdots \otimes u_n) \otimes (v_1 \otimes \cdots \otimes v_n) \mapsto \langle u_1, v_{w(1)}, \rangle \cdots \langle u_n, v_{w(n)} \rangle$$

for $w \in S_n$.

For example,

$$\mathrm{Hom}_{GL(V)}(V^{\otimes 2} \otimes (V^{\vee})^{\otimes 2}, \mathbb{C})$$

has a basis

$$u_1 \otimes u_2 \otimes v_1 \otimes v_2 \to \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle$$
$$u_1 \otimes u_2 \otimes v_1 \otimes v_2 \to \langle u_1, v_2 \rangle \langle u_2, v_1 \rangle$$

See the handout posted on the course page for more explanation.

For SL_2 and SL_3 we have "nice" descriptions of $\mathrm{Hom}_{SL}(V^{\otimes n}, V^{\otimes m})$. What makes them "nice"?

(1) We have an explicit basis.
(2) There is a good description of the composition map
(3) Our "good" basis has a lot of symmetry.