

THE PERFECT MATCHING BASIS FOR $SL(2)$ -INVARIANTS

CHRIS FRASER

A reference for today's talk is Greg Kuperberg's *Spiders for Rank 2 Lie Algebras*, available here: <http://arxiv.org/abs/q-alg/9712003>.

Today we'll begin discussing webs, spiders etc. These provide an excellent tool for thinking about $(V^{\otimes 2n})^{SL_n}$ in low dimensions. People have still not satisfactorily extended these things to higher dimensions. For example, Kuperberg's paper discusses the rank 2, SL_3 case. In today's talk, we'll do the SL_2 case.

Let $V \cong \mathbb{C}^2$ be the defining representation of GL_2 . Our goal will be to find a "nice" basis for

$$(V^{\otimes 2n})^{SL_2},$$

the SL_2 invariants of $V^{\otimes 2n}$.

Equivalently, these are the homomorphisms

$$\text{Hom}_{SL_2}(V^{\otimes n}, V^{\otimes n}).$$

(Recall from the previous lecture that since $\dim V = 2$, we have that $V^\vee = \det^{-1} \otimes V = V$ as $GL_2(V)$ -reps, and hence $V^\vee = V$ as $SL_2(V)$ -reps. So we are not being very careful about duals in this lecture).

We need to specify what we mean for a basis to be nice. A good starting point is that the basis exhibits LOTS OF SYMMETRY. The representation theory of $(V^{\otimes 2n})^{SL_2}$ should carry with it a decent amount of symmetry, e.g. perhaps from permuting tensor factors ($v \otimes w \mapsto w \otimes v$). We already know our space has a basis indexed by SYT of shape (n, n) , but there aren't a lot of obvious operations we can perform on such tableaux to reflect the symmetries of our problem. One operation we *do* have was mentioned last class: we reverse the relative sizes of the entries, and then rotate the diagram 180 degrees. This symmetry sends

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 6 & 5 & 3 \\ \hline 4 & 2 & 1 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array}$$

and

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 6 & 5 & 2 \\ \hline 4 & 2 & 1 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}$$

However, this is the only obvious symmetry we have lying in front of us. It is true that there is another operation on SYT called promotion. This is a little more involved and David will discuss it later. For now, we're not going to talk about it.

Question from the field: Can you remind me why the SYT of shape (n, n) are a basis.

Answer: Think of $(V^{\otimes 2n})^{SL_2}$ as the component of $\mathbb{C}[x_{ij}]^{SL_2}$, $1 \leq i \leq 2, 1 \leq j \leq 2n$ of degree 1 in each column (with each copy of V representing a column). The (First) Fundamental Theorem of Invariant Theory for SL_2 says that such invariants are generated by all 2×2 determinants. Recall that the GL_{2n} -rep $V_{(n,n)}(2n)$ is spanned by all n -fold products of 2×2 top-justified determinants in $\mathbb{C}[x_{ij}]$, $1 \leq i, j \leq 2n$, so it actually lands in $\mathbb{C}[x_{ij}]$, $1 \leq i \leq 2, 1 \leq j \leq 2n$. Our space of SL_2 invariants is therefore just the component of $V_{(n,n)}(2n)$ of degree 1 in each column. This GL_{2n} -rep has a basis of semistandard Young tableaux of shape (n, n) , and if i appears k times in the SSYT, the corresponding element in $V_{(n,n)}(2n)$ will be of degree k in the i th column. Therefore, the component of $V_{(n,n)}(2n)$ of degree 1 in each column corresponds to SSYT where each entry $1, 2, \dots, 2n$ is used exactly once, which are standard Young tableaux.

The “noncrossing-matching basis” we are about to discuss is more symmetric than the SYT-basis. HOWEVER it is too much to ask for our basis to allow symmetries corresponding to arbitrary permutations of tensor factors. There is a good justification of this: if we want our basis to extend to the setting of quantum group representation theory, then arbitrary permutations of tensor factors are no longer SL_2 -equivariant. On the other hand, we might want to restrict ourselves to merely cyclic permutations of tensor factors, for example $x \otimes y \otimes z \mapsto y \otimes z \otimes x$. It turns out these permutations *do* remain SL_2 -equivariant, and we would like our basis to exhibit this type of symmetry.

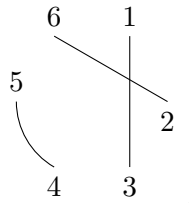
Added by David Another reason why it is unreasonable to hope that our basis has symmetries for arbitrary permutations of tensor factors is that the action of S_{2n} on the invariants is usually NOT a permutation action (nor some trivial modification, such as permutation tensor sign). For example, when $n = 3$, the representation of S_6 is the Specht module $Sp(\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$ which we will meet next week, and this simply is not a permutation representation.

1. NON-CROSSING MATCHINGS

On Problem Set 7, we will work out a bijection between SYT of shape (n, n) and non-crossing matchings. Recall, a **Non-crossing Matching** on $2n$ elements is a way to pair them so that no two matchings cross. The number of noncrossing matchings is the n^{th} Catalan number, $\frac{1}{n+1} \binom{2n}{n}$. Here’s an example of a noncrossing matching on 6 elements, which we will use as a running example for the rest of today.



Here is a second example of a matching, but it is not noncrossing, because the line between 1 and 3 crosses the line between 6 and 2.

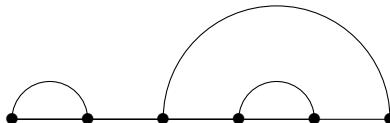


We will not consider such matchings.

Our bijection leads us to a new idea: is it possible to make the non-crossing matchings into a basis for $(V^{\otimes 2n})^{SL_2}$ in a “good” way. In particular, note that these noncrossing matchings exhibit the cyclic symmetry we were seeking above (since rotating a matching doesn’t change whether or not it is a “crossing” matching).

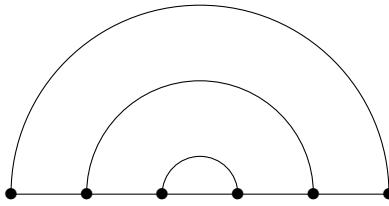
Some conventions about matchings:

- We will call these noncrossing matchings “webs”; it’s a good short word to replace non-crossing matching, and we will use the same word later, when we move up to rank 2.
- We will often draw matchings on a horizontal line, rather than on a circle (nothing is lost in doing this). The matching in (1) is drawn below.

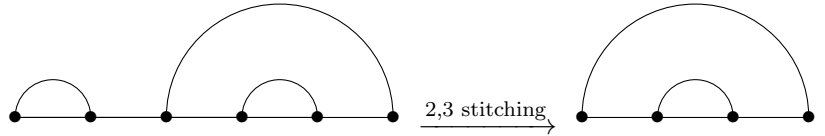


- Let R denote counterclockwise rotation of the matching. For example, the pairing in our running example has the pairs $\{1, 2\}\{3, 6\}\{4, 5\}$ and rotated pairing has pairs $\{6, 1\}, \{2, 5\}, \{3, 4\}$.

The horizontal diagram of the rotated pair is given by



- We have a notion of **join** of two noncrossing matchings, which is just concatenation. For example, our running example is the join of the matching $\{1, 2\}$ with the matching $\{1, 4\}\{2, 3\}$. If our matchings have size $2n$ and $2m$ respectively, the result will be a matching of size $2(n + m)$. Ultimately, we'd like the join operation to correspond to taking \otimes of a basis element from $V^{\otimes 2n}$ with $V^{\otimes 2m}$, which lands in $V^{\otimes 2(n+m)}$.
- We also have a notion of **stitching**. Given a noncrossing matching, we can choose two adjacent numbers and “identify” them, thereby obtaining a matching on two fewer elements. For example, stitching the numbers 2 and 3 together in our running example gives the noncrossing matching given by $\{1, 4\}, \{2, 3\}$. The picture is



We only allow ourselves to stitch adjacent numbers, since otherwise we might not preserve the property of being non-crossing. In terms of our basis, we would like this stitching operation to coincide with contraction, which comes from

$$V \times V \cong V \times V^\vee \rightarrow \mathbb{C}.$$

Note that if we identify the numbers 4 and 5 in our running example, we get a loop at the identified point. This should represent a scalar. We will see what happens in this case later on.

An aside: What is the isomorphism $V \cong V^\vee$. Ie, if $v^\vee \in V^\vee$ what v does v^\vee correspond to? Our answer is that v is the vector such that $\langle v^\vee, \bullet \rangle = \det(\bullet, v)$, where the \bullet symbol is supposed to represent the place you stick in vectors. Explicitly, suppose e_1, e_2 are a basis for V and e^1, e^2 are the corresponding dual basis. Then $e^1 \mapsto -e_2$ and $e^2 \mapsto e_1$. This is a particular case of the *Hodge star* map, which you may have heard of elsewhere.

2. CONSTRUCTING THE NONCROSSING MATCHING BASIS

We want maps

$$\varphi_{2n} : \{\text{Webs on } 2n \text{ elts}\} \rightarrow (V^{\otimes 2n})^{SL_2}$$

respecting the operations on matchings we defined in the previous section. Explicitly:

- (1) *Joining webs should correspond to* \otimes . That is

$$\varphi_{2(n+m)}(W_1, W_2) = \varphi_{2n}(W_1) \otimes \varphi_{2m}(W_2),$$

for any two noncrossing matchings W_1, W_2 .

- (2) Rotations of webs should correspond to the cyclic permutation of tensor factors, with a minus sign. The minus sign is unfortunate, but we'll see in a second there's not any way of getting around it. We will abuse notation and use R for both counterclockwise rotation of webs and cyclic permutation left of tensor factors; hence we want

$$\varphi_{2n}(R(W)) = -R(\varphi_{2n}(W)).$$

- (3) Stitches of webs should correspond to contraction of the appropriate tensor factors.
- (4) Once we take spans, φ_{2n} should extend to an isomorphism between the formal complex span of the webs, and $(V^{\otimes 2n})^{SL_2}$.

From now on, we'll drop subscripts on φ . The properties above will nearly force our construction.

Note that $(V^{\otimes 2})^{\text{SL}_2}$ is one-dimensional, spanned by $e_1 \otimes e_2 - e_2 \otimes e_1$. This corresponds to the determinant map. We will denote this element of $(V^{\otimes 2})^{\text{SL}_2}$ by a .

There is only one web on two elements, so we may as well make φ send this web to a . Rotating this web does nothing, but rotating a gives $-a$. There isn't really a way around this, and it justifies the minus sign we had in property (2) above.

Now notice that every web is uniquely constructed from the web on two points by taking joins and by applying rotations in a certain way. Given a web W on $2n$ vertices where $n \geq 2$, if 1 is not paired with $2n$ (so if there is no single arc going over all of the other arcs in the picture), then the web is already a join of two smaller webs. Otherwise, one sees that

$$R^{-1}(W)$$

is the join of the web on two vertices with a smaller web on $2(n-1)$ vertices. For example,

Thus, properties 1) and 2) above, along with our choice that $\varphi(\text{arc}) = a$, provides a definition φ for every web: φ sends joins to \otimes , and satisfies

$$R(\text{arc}, **) \mapsto_{\varphi} -R(a \otimes \varphi(**)).$$

where $**$ indicates a web whose image under ϕ we already know.

Since restricting ourselves to this limited use of R allows us to build each web in a unique way, this gives a well-defined map ϕ . Next class we'll prove that this definition behaves well with respect to multiple rotations, and with respect to stitching.

Right now we'll check that stitching behaves well in one example, using the explicit formula for contraction we found in the "aside" above.

Namely, we have specified that

$$\varphi(\text{two arcs}) = a \otimes a.$$

Stitching together along 2 and 3 gives us the unique web on two points (which is sent to a by φ). For this to work out, we want to make sure that contracting $a \otimes a$ along the second and third tensor factor gives us a . Note that

$$a \otimes a = e_1 \otimes e_2 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 \otimes e_1$$

By the aside, $e_2 \otimes e_1 \mapsto \langle e^2, e_2 \rangle = 1$. Similarly, one checks that $e_2 \otimes e_2$ contracts to 0, $e_1 \otimes e_1$ contracts to 0, and $e_1 \otimes e_2$ contracts to -1 , via

$$e_1 \otimes e_2 \mapsto \langle -e^1, e_1 \rangle = -1.$$

Hence, the contraction of $a \otimes a$ in the second and third tensor factors is

$$e_1 \otimes e_2 - e_2 \otimes e_1,$$

which is a , as desired.

Kevin adds: We're also ready to see what scalar the loop corresponds to. This small bit got lost in time and space between this lecture and the next, but it's an easy calculation using stitching (where I will denote stitching and contraction by S_1):

$$\varphi(\text{loop}) = \varphi(S_1(\text{two arcs})) = S_1(e_1 \otimes e_2 - e_2 \otimes e_1) = \langle -e^1, e_1 \rangle - \langle e^2, e_2 \rangle = -2.$$

Hence we see that a loop should be the scalar -2 .