MORE CHARACTERS AND THE RING OF MATRIX COEFFICIENTS – NOTES FOR OCTOBER 3

SCRIBE: CHRIS FRASER

1. More On Characters

Last time, we introduced characters. Namely if $\rho\colon G \to \mathrm{GL}(V)$ is a representation, then the map

$$\chi_V \colon G \to \mathbb{C}$$

sending $g \mapsto \mathrm{Trace}(\rho(g))$ is called the character of V.

Note that $\chi(g) = \chi(g')$ whenever g and g' are in the same conjugacy class. Indeed, if $g' = hgh^{-1}$

$$\begin{aligned} \chi_V(hgh^{-1}) &= \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) \\ &= \mathrm{Tr}(\rho(g)) \\ &= \rho(g) \end{aligned}$$

where we have used above that trace is well-defined on conjugacy classes. We call the character a ***class function***, since it is constant on conjugacy classes.

Here are some useful properties of characters:

- $\chi_{\mathbb{C}}(g) = 1$, where $\mathbb{C}$ is the trivial representation.
- $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$.
  This is true because each $\rho(g)$ will have a block-diagonal form, where the first block describes the action of g on V, and the second block of describes the action of g on W. The trace of the whole matrix will be the sum of the traces on the individual blocks.
- $\chi_{V\otimes W} = \chi_V(g)\chi_W(g)$.
  This is similar, pick bases $\{e_i\}_i$ for V and $\{f_j\}_j$ for W. Then $\{e_i \otimes f_j\}_{i,j}$ is a basis for $V \otimes W$, and the coefficient of $e_i \otimes f_j$ inside $g \cdot e_i \otimes f_j$ will be the product of the coefficient of e_i in $g \cdot e_i$ times the ceofficient of f_j in $g \cdot f_j$.
- $\chi_{V^\vee}(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)}$ where the last equality is only when G is compact.

Specializing to the case that G is compact, we showed last time that

$$\int_G \chi_V(g)dg = \mathrm{dim}V^G$$

and

$$< \chi_V, \chi_W >:= \int_G \overline{\chi_W(g)}\chi_V(g)dg = \dim \mathrm{Hom}_G(V,W).$$

The two facts above tell us a lot:

- Characters of simple representations are orthonormal inside the vector space of functions on G, with respect to the Hermitian Inner Product defined in Corollary 7 of the previous lecture. This follows from the second fact above and Schur's Lemma (which describes $\mathrm{Hom}(V,W)$ when V,W are both simples).
- A representation is determined by its character. This is Corollary 9 in the previous lecture.

As was mentioned in class, the second fact above shows that $\dim \mathrm{Hom}$ is symmetric in its arguments, although we already knew this from the previous lecture when we described $\mathrm{Hom}(V,W)$ as a direct of sum of matrix algebras (using decomposition into irreducibles).

Let's proceed. Last class, we defined

$$\pi = \int_G \rho(g)dg,$$

which is an element of $\mathrm{End}(V)$. Explicitly, its action on a $v \in V$ is given by

$$\pi \cdot v = \int_G \rho(g) \cdot v dg.$$

From last class, the kernel of this is some G-subrepresentation of V, and $V = \ker(\pi) \oplus V^G$, so that π is the projection onto the G-invariants of V.

We will generalize this now. Let V be any representation and let U be a simple representation. We define

$$\pi_U := \dim V \int_G \overline{\chi_U(g)} \rho_V(g) dg,$$

which we will think of as an element of $\mathrm{End}(V)$.

This is like our definition of π previously, but "weighted" by the character of U (the definition we gave above corresponds to the trivial representation).

Now we check that for any U, $\pi_U \in \mathrm{Hom}_G(V, V)$, which is to say it commutes with the g action. Indeed:

$$\begin{aligned}
\pi_U \rho_v(h) &= \dim V \int \overline{\chi_U(g)} \rho_v(g) \rho_v(h) dg \\
&= \dim V \rho_V(h) \int \overline{\chi_U(g)} \rho_V(h^{-1} g h) dg \\
&= \dim V \rho(h) \int \overline{\chi_U(h^{-1} g' h)} \rho_V(g') dg \\
&= \dim V \rho(h) \int \overline{\chi_U(g')} \rho_V(g') dg \\
&= \rho(h) \pi_U
\end{aligned}$$

Furthermore the formula for π_U shows it takes each simple to itself: it "looks like"

$$\pi_U = (\text{scalar}) \int (\text{scalar}) \rho_V(g) dg,$$

and since every $\rho(g)$ fixes subrepresentations, so will π_U. Hence, $\pi_U|_W$ is in $\mathrm{End}_G(W)$ for each simple W showing up in V. Thus by Schur's Lemma, if W is a simple representation, π_U must be a scalar times the identity. What scalar does π_U act by?

On W:

$$\begin{aligned}
\mathrm{Tr}(\pi_U) &= \dim W \int \overline{\chi_U(g)} \mathrm{Tr} \rho_W(g) dg \\
&= \dim W \int \overline{\chi_U(g)} \chi_W(g) dg.
\end{aligned}$$

Now the integral on the RHS is 1 or 0 according to whether $U \cong W$ or not.

Let π_U act on W by the scalar a. So $Tr \pi_U = a \dim W$. We see that a is either 1 or 0 depending on whether or not $U \cong W$.

Conclusion: π_U projects onto U-isotypic component in a G-equivariant way.

2. Matrix Coefficients

We want to understand complex-valued functions on G in terms of representations, but we're not analysts. And even analysts don't care about all functions! We have to restrict the functions we are going to think about.

From now on, let G be a topological group and $C^0(G)$ be the set of continuous functions $G \to \mathbb{C}$. Note $G \times G$ acts on $C^0(G)$ by

$$((g_1, g_2) \cdot f)(h) = f(g_1^{-1} h g_2).$$

Here's a construction that will suggest the class of functions on G that we will make statements about.

Let V be a finite-dimensional continuous representation of G. Let $\lambda\colon \mathrm{End}(V) \to \mathbb{C}$ be $\mathbb{C}$-linear. Then $\lambda(\rho_V(g))$ is a continuous function $G \to \mathbb{C}$. Such a function is called a ***matrix coefficient***.

Why? Because if $\lambda\colon \mathrm{End}(V) \to \mathbb{C}$ is the functional that picks out a matrix entry, than $\lambda(\rho_V(g))$ picks out that matrix entry of the matrix that describes g's action on V. We will let $\mathcal{O}$ denote the set of matrix coefficients.

The reverse is not quite true (not every "matrix coefficient" is picking out a matrix entry of g). For example, take λ to be the trace function, which is a perfectly acceptable map $\mathrm{End}(V) \to \mathbb{C}$. It is not possible to change bases so that this is actually a matrix entry. If you had a sufficiently abstract introduction to linear algebra, the linear functionals that *are* just matrix entries are the rank 1 tensors in $\mathrm{End}(V) = V \otimes V^\vee$.

Theorem 1. *Let $f \in C^0(G)$. TFAE:*

1. *Span $(g_1, g_2) \cdot f$ is finite dimensional.*
2. *Span $(g, 1) \cdot f$ is finite dimensional.*
3. *Span $(1, g) \cdot f$ is finite dimensional.*
4. *f is a matrix coefficient.*

Clearly, (1) implies either of (2) or (3). It is also relatively easy to see that (4) implies (1). Take f to be a matrix coefficient associated to $\lambda \in \mathrm{End}(V)$. Then

$$\begin{aligned}(g_1, g_2) \cdot f(h) &= \lambda(g_1^{-1}\rho_V(h)g_2) \\ &= \lambda' \circ \rho_V(h)\end{aligned}$$

where λ' is another element of $\mathrm{End}(V)$. (Explictly, $\lambda'(A) = \lambda(g_1^{-1}Ag_2)$.). Hence, $(g_1, g_2) \cdot f$ is contained in the finite dimensional vector space of matrix coefficients.

We'll see the remaining "hard" implication (2) implies (1), next class. For next time, a good exercise would be to think about why $\mathcal{O}(G)$ is a ring.