# $GL_n$ REPRESENTATION THEORY NOTES FOR 10-31

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We continue our study of web diagrams and non-crossing matchings.

**Recall**: We were in the process of showing that the non-crossing matchings are a basis for $(V^{\otimes 2n})^{\mathrm{SL}_2}$, where $V = \mathbb{C}^2$ is the defining representation for $\mathrm{GL}_2(\mathbb{C})$.

Important notation:

- Recall that $\frown$ is the web/non-crossing matching with two elements. We will write $\overset{\frown}{w}$ to denote a web $w$ enclosed by $\frown$.
- If $w$ is a web, let $R(w)$ be the web obtained by rotating the picture counterclockwise one step.
- If $v$ is a pure tensor, let $R(v)$ be the pure tensor obtained by rotating the entries "left". For example, $R(x \otimes y \otimes z) = y \otimes z \otimes x$.
- If $w$ is a web and $i$ an index, $S_i(w)$ is the web obtained by "stitching" $i$ to $i+1$.
- If $v = \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots$ is a pure tensor (and $i$ an index), then $S_i(v)$ is the pure tensor of length 2 shorter, obtained by contracting $v_i \otimes v_{i+1}$ to the scalar $\det(v_{i+1}, v_i)$.

## 1. The main theorem

We prove the following.

**Theorem 1.** *Let $W_{2n}$ be the vector space on the set of webs on $2n$ elements. Then there is a (family of) isomorphism(s)*

$$\varphi : W_{2n} \to (V^{\otimes 2n})^{\mathrm{SL}_2},$$

*with the following properties:*

- *(Rotation) For any web $w$, $\varphi(R(w)) = -R(\varphi(w))$.*
- *(Stitching/Contraction) For any web $w$ and $i$, $\varphi(S_i(w)) = S_i(\varphi(w))$.*
- *(Concatenation/Multiplication) For any webs $w_1, w_2$, $\varphi(w_1 w_2) = \varphi(w_1) \otimes \varphi(w_2)$.*

**Remark**. Last time, we showed that any web can be built uniquely out of copies of the basic web $\frown$ by composing joins and the specific rotation $R(\frown \cdot)$. This gives a "standard" way to build a web, so we do have a well-defined map $\varphi$. We still need to show that $\varphi$ behaves correctly with arbitrary joins, rotations and contractions.

*Proof.* First of all, joins work by construction. For rotations, we need to show the identity

$$\varphi(R(\overset{\frown}{w_1} w_2)) = -R(\varphi(\overset{\frown}{w_1} w_2)).$$

We consider the left-hand side: after applying the rotation, this is just

$$\begin{aligned} \varphi(w_1 \overset{\frown}{w_2}) &= \varphi(w_1) \otimes \varphi(\overset{\frown}{w_2}) \\ &= \varphi(w_1) \otimes \varphi(R(\frown w_2)) \\ &= \varphi(w_1) \otimes -R(a \otimes \varphi(w_2)), \end{aligned}$$

where $a = \varphi(\frown) = e_1 \otimes e_2 - e_2 \otimes e_1$ is from our initial definition of $\varphi$. We expand this and rearrange:

$$\begin{aligned}
&= \varphi(w_1) \otimes -R\big(e_1 \otimes e_2 \otimes \varphi(w_2) - e_2 \otimes e_1 \otimes \varphi(w_2)\big) \\
&= -\varphi(w_1) \otimes \big(e_2 \otimes \varphi(w_2) \otimes e_1 - e_1 \otimes \varphi(w_2) \otimes e_2\big) \\
&= -R\big(e_1 \otimes \varphi(w_1) \otimes e_2 \otimes \varphi(w_2) - e_2 \otimes \varphi(w_1) \otimes e_1 \otimes \varphi(w_2)\big) \\
&= -R\Big(R\big(e_2 \otimes e_1 \otimes \varphi(w_1) - e_1 \otimes e_2 \otimes \varphi(w_1)\big) \otimes \varphi(w_2)\Big) \\
&= R\big(R(a \otimes \varphi(w_1)) \otimes \varphi(w_2)\big) \\
&= -R\big(\varphi(R(\frown w_1)w_2)\big) \\
&= -R(\varphi(\overset{\frown}{w_1}\, w_2).
\end{aligned}$$

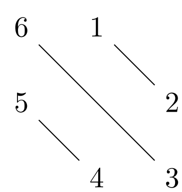
This confirms that rotations work as intended. For stitchings, we omit the proof, though the computation is similar. (It's easier to start with the right hand side, $S_i(\varphi(w))$, since it's easier to transform the larger tensor into a smaller one by contracting two tensor factors into a scalar.)

We show that $\varphi$ is an isomorphism. We know the dimensions are correct (by a homework problem), so we need only check injectivity or surjectivity. We'll show surjectivity. We think of $(V^{\otimes 2n})^{\mathrm{SL}_2}$ as $\mathrm{Hom}_{SL_2}(V^{\otimes 2n}, \mathbb{C})$, which in turn we think of as $SL_2$-invariant multilinear functions $V \times \cdots \times V \to \mathbb{C}$.

We have a spanning set

$$(v_1, \ldots, v_n) \mapsto \det(v_{w(1),w(2)}) \cdots \det(v_{w(2n-1),w(2n)}), \qquad w \in S_{2n}.$$

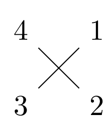
It suffices to show that all of these are in $\varphi(W_{2n})$. In this language, for example,




corresponds to the map

$$(v_1, \ldots, v_6) \mapsto \det(v_1, v_2)\det(v_3, v_6)\det(v_4, v_5) = \Delta_{12}\Delta_{36}\Delta_{45}.$$

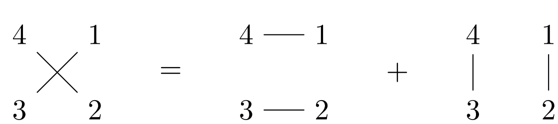
If we allow crossings, then we certainly span the whole space. So, it suffices to express crossings like




in terms of webs. To do this, we expand using the Plücker relation,

$$\Delta_{13}\Delta_{24} = \Delta_{14}\Delta_{23} + \Delta_{12}\Delta_{34},$$

which says, in terms of web diagrams,




This completes the proof. ∎

## 2. Generalizing to $SL_3$

We can still use webs to get a basis for $(V^{\otimes 3n})^{SL_3}$. Various things have changed, though (and in fact it's not known how to find a web basis if we instead try $SL_4$). In particular:

- The vector space $V = \mathbb{C}^3$ is now the defining representation of $SL_3$ (and $GL_3$),
- It is no longer self-dual(!): $V \ncong V^\vee$. However,
- We do have $V^\vee \cong \bigwedge^2 V$. Indeed, for characters,

$$x_1^{-1} + x_2^{-1} + x_3^{-1} = (x_1x_2x_3)^{-1}(x_1x_2 + x_1x_3 + x_2x_3),$$

which shows that

$$V^\vee \cong (\det)^{-1} \otimes \bigwedge^2 V \text{ as GL}_3\text{-representations},$$

hence $V^\vee \cong \bigwedge^2 V$ for $SL_3$-representations. Explicitly, we have the correspondence

$$\bigwedge^2 V \ni v_1 \wedge v_2 \longleftrightarrow \det(v_1, v_2, \ \cdot\ ) \in V^\vee.$$

As above, we'll think of $(V^{\otimes 3n})^{SL_3}$ as the vector space of multilinear maps

$$\underbrace{V \times \cdots \times V}_{3n} \to \mathbb{C}.$$

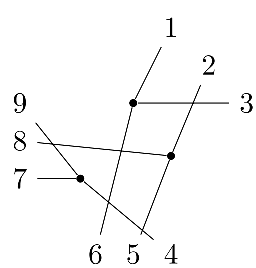
We have a basis consisting of $3 \times n$ standard Young tableaux:

| | | |
|---|---|---|
| 1 | 2 | 4 |
| 3 | 5 | 7 |
| 6 | 8 | 9 |

corresponds to $\Delta_{136}\Delta_{258}\Delta_{479}$. This is good, but does not have cyclic symmetry: if we rotated $1 \to 9 \to 8 \to \cdots$, we would end up with

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 9 | 8 |

which is no longer a standard Young tableau. That said, we can draw our SYT as a "tripod diagram":



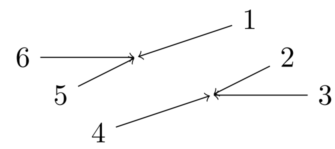

which has some rotational symmetry. There are various problems with this approach, however. For one thing, arbitrary "tripod diagrams" are too numerous, and don't correspond nicely to SYTs. Moreover, the "totally noncrossing" versions of these tripod pictures are too *few* to span the whole space (and they still don't correspond neatly to SYTs). For example, there are 10 "tripod diagrams" on 6 points, 5 SYTs of shape $(2, 2, 2)$, and only 3 "totally noncrossing tripod diagrams".

### 2.1. **Tensor Diagrams.** One solution is to use tensor diagrams, defined as follows.

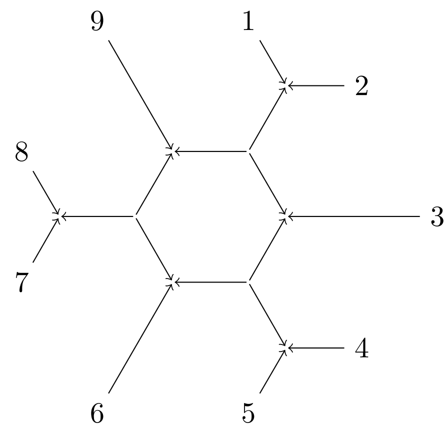
**Definition**. A ***tensor diagram*** for $(V^{\otimes 3n})^{SL_3}$ is a directed graph in a disk, such that

- There are $3n$ boundary sources of degree 1,
- All interior vertices are *either* sources *or* sinks, and have degree 3.

For example,




corresponds to the product of determinants $\Delta_{234}\Delta_{156}$. However, more complicated diagrams may not clearly correspond to anything 'nice' (or at least anything that we already have a name for):




Next class, we'll describe how these correspond to invariants, and we'll see that these give a good description for $SL_3$ invariants.