NOTES FOR OCT 05

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Today we will prove a weak version of the Peter-Weyl Theorem.

1. Notation

Through out these notes, G will be a topological group, $C^0(G) = \{\text{continuous functions } G \to \mathbb{C}\}$, and $G \times G$ acts on $\phi \in C^0(G)$ by $(g_1, g_2) \cdot \phi(h) = \phi(g_1^{-1} h g_2)$

Last time we begin the proof of the following proposition: **Proposition**: For $\phi \in C^0(G)$, the following are equivalent

(1) $\text{Span}(g_1, g_2) \cdot \phi$ is finite dimensional
(1′) $\text{Span}(g, 1) \cdot \phi$ is finite dimensional
(1″) $\text{Span}(1, g) \cdot \phi$ is finite dimensional
(2) $\phi = \lambda(\rho_V(g))$ for some finite dimensional continuous G-representation V and some $\lambda : \text{End}(V) \to \mathbb{C}$

Functions $\phi \in C^0(G)$ satisfying these properties are called matrix coefficients.

From last time, we have that $(2) \Rightarrow (1) \Rightarrow (1'), (1'')$. All that is left is to prove $(1'') \Rightarrow (2)$, and $(1') \Rightarrow (2)$ will follow analogously.

Set $V = \text{Span}(1, g) \cdot \phi = \text{Span}\{h \mapsto \phi(hg)\}$. We need to check that V is a continuous G representation. Let $\phi_1, \cdots, \phi_n$ be a basis of V. For $h \in G$, let ϵ_h be evaluation at h (i.e. ϵ_h is the map $\phi \mapsto \phi(h)$). As h runs over G, the ϵ_h span $V^\vee$.

Let $\epsilon_{h_1}, \cdots, \epsilon_{h_n}$ be a basis of $V^\vee$. The matrix $(\epsilon_{h_i}(\phi_j))$ is invertible, since we obtained it by pairing a basis for V against a basis for $V^\vee$. For $\psi \in V^\vee$, we have

$$\psi = (\phi_1 \cdots \phi_n)(\epsilon_{h_i}(\phi_j))^{-1} \begin{pmatrix} \psi(h_1) \\ \vdots \\ \psi(h_n) \end{pmatrix}$$

so

$$\rho(g) \cdot \psi = (\phi_1 \cdots \phi_n)(\epsilon_{h_i}(\phi_j))^{-1} \begin{pmatrix} \psi(h_1 g) \\ \vdots \\ \psi(h_n g) \end{pmatrix},$$

which is continuous in g.

So we have a continuous representation V. Let $\lambda : \text{End}(V) \to \mathbb{C}$ be the map $\alpha \mapsto \epsilon_{\text{Id}}(\alpha(\phi))$. Then $\lambda(\rho_V(g)) = \phi(g)$, so λ is the desired map.

2. The ring of matrix coefficients

We define $\mathcal{O}(G)$, the ring of matrix coefficients, to be $\{\phi \in C^0(G) : \phi \text{ is a matrix coefficient of } V\}$. This forms a ring since if $\phi, \psi \in \mathcal{O}(G)$, then by the proposition $(G \times G) \cdot \phi = \text{Span}(\phi_1, \cdots, \phi_m)$ and $(G \times G) \cdot \psi = \text{Span}(\psi_1, \cdots, \psi_n)$ for some $\phi_1, \cdots, \phi_m, \psi_1, \cdots, \psi_n \in C^0(G)$. Then $(G \times G) \cdot (\phi + \psi) \subset \text{Span}(\phi_1, \cdots, \phi_m, \psi_1, \cdots, \psi_n)$ and $(G \times G) \cdot (\phi\psi) \subset \text{Span}(\phi_i \psi_j)$ so both $\phi + \psi$ and $\phi\psi$ are matrix coefficients by the proposition.

3. The Weak Peter Weyl Theorem

Let G be a compact group. **Theorem:**

$$\mathcal{O}(G) \cong \bigoplus \text{End}(V)^\vee \text{ as } G \times G \text{ representations.}$$

Here the direct sum is over all isomorphism classes of irreducible representations of G, each listed once. In addition, $\mathrm{End}(V)^\vee$ is orthogonal to $\mathrm{End}(W)^\vee$ under $< \phi, \psi >= \int \phi(g)\overline{\psi(g)}dg$

Lemma: If V and W are simple G and H representations then $V \otimes W$ is a simple $G \times H$ representation.

We will prove this for when G and H are compact, though this is true for any G and H. We have

$$\int_{G\times H} \chi_{V\otimes W}(g,h)\overline{\chi_{V\otimes W}(g,h)}d(g,h) = \int_G \chi_V(g)\overline{\chi_V(g)}dg \int_G \chi_W(h)\overline{\chi_W(h)}dh = 1 \cdot 1 = 1,$$

so $V \otimes W$ is simple.

4. Proof of the Weak Peter-Weyl Theorem

We have a natural map $\bigoplus \mathrm{End}(V)^\vee \to \mathcal{O}(G)$. We want to show that this is an isomorphism.

Injectivity: This is a map of $G \times G$ representations, so the kernel is a $G \times G$ subrepresentation. This means that the kernel must be of the form $\bigoplus_{V\in S} \mathrm{End}(V)^\vee$. $\mathrm{End}(V)^\vee \to \mathcal{O}(G)$ is not the zero map for any irreducible V though, so the kernel must be 0.

Surjectivity: Let $\phi = \lambda(\rho_W(g))$ for some $W \cong \bigoplus V_i$. $\mathrm{End}(W) = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$ so $\lambda = \sum \lambda_{ij}$ for some $\lambda_{ij} : \mathrm{Hom}(V_i, V_j) \to \mathbb{C}$. But $\rho_W(G) \subset \bigoplus \mathrm{End}(V_i)$ so $\lambda(\rho_W(g)) = \lambda'(\rho_W(g))$ where $\lambda' = \sum \lambda_{ii}$. For V an irreducible representation, let $\lambda_V = \sum_{V_i \cong V} \lambda_{ii}$, so $\lambda'(\rho_W(g)) = \sum_V \lambda_V(\rho_V(g))$, which is in the image of $\bigoplus \mathrm{End}(V)^\vee$.

Orthogonality This was done pretty badly in class; see the next lecture's notes for a better presentation. (Statement added by David.)