NOTES FOR OCTOBER 8

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1. Matrix coefficients are orthogonal

Recall from last time: we let G be a compact group, and let $\mathcal{O}(G)$ the ring of matrix coefficients. Then the Weak Peter-Weyl Theorem says

$$\mathcal{O}(G) = \bigoplus_{V \text{ an isomorphism class of irreps}} V^\vee \otimes V$$

as a $G \times G$ representation, and the decomposition is orthogonal.

Remember how the embedding of $V^\vee \otimes V$ into $\mathcal{O}(G)$ works. We have $V^\vee \otimes V \cong \operatorname{End}(V)^\vee$. Given a linear functional $\lambda : \operatorname{End}(V) \to \mathbb{C}$, the isomorphism carries λ to the function $\lambda(\rho_V(g))$ on G. We still need to prove orthogonality.

Let V, W be nonisomorphic simples, and let $\lambda \in \operatorname{End}(V)^\vee$ and $\mu \in \operatorname{End}(W)^\vee$. We want to compute

$$\int_G \lambda(\rho_V(g))\overline{\mu(\rho_W(g))}dg$$

The action of G on W preserves a positive definite Hermitian form, so we can choose bases where the action of W is unitary. That is,

$$\overline{\rho_W(g))} = \rho_W(g)^{-T} = \rho_{W^\vee}(g)$$

using the dual coordinates on $W^\vee$. So we can rewrite the above expression as

$$\int_G \lambda(\rho_V(g))\mu'(\rho_{W^\vee}(g))dg = \int_G (\lambda \otimes \mu')(\rho_{V \otimes W^\vee}(g))dg = (\lambda \otimes \mu')\int_G \rho_{V \otimes W^\vee}(g)dg$$

where $\lambda \otimes \mu' \in \operatorname{End}(V \otimes W^\vee)^\vee$ and $\rho_{V \otimes W^\vee}(g) \in \operatorname{End}(V \otimes W^\vee)$.

Since V and W are non-isomorphic simples, we have

$$(V \otimes W^\vee)^G = \operatorname{Hom}(W, V)^G = \operatorname{Hom}_G(W, V) = 0$$

and so

$$(\lambda \otimes \mu')\int_G \rho_{V \otimes W^\vee}(g)dg = (\lambda \otimes \mu')(0) = 0$$

This completes the proof.

2. An aside

Question from the floor: Is it obvious that $(U \otimes V)^G = (V \otimes U)^G$? Can you talk about more about group actions on tensor products?

Sure! We have $U \otimes V \cong V \otimes U$ as G-representations.

The diagonal map gives an embedding of G into $G \times G$, and we have

$$G \hookrightarrow G \times G \circlearrowright U \otimes V$$

$\operatorname{End}(V)$ has a natural bilinear form $\langle A, B \rangle = \operatorname{Tr}(AB)$. This gives a natural isomorphism $\operatorname{End}(V) \to \operatorname{End}(V)^\vee$. This is not an isomorphism of $G \times G$ representations since $\langle\,,\,\rangle$ is not G-invariant.

3. Consequences of Peter-Weyl

Consequence 1: If W is a finite dimensional $G \times G$ subrepresentation of $C^0(G)$, then

$$W \cong \bigoplus_{V \in S} V^\vee \otimes V$$

for some set S of nonisomorphic simples.

Consequence 2: (Maschke's Theorem) If $V_1, V_2, \ldots, V_k$ are pairwise isomorphic simples, then the image of G spans

$$\mathrm{End}(V_1) \oplus \mathrm{End}(V_2) \oplus \cdots \oplus \mathrm{End}(V_k)$$

Proof: Suppose $\lambda : \oplus \mathrm{End}(V) \to \mathbb{C}$ vanishes on the image of G. Then $\lambda = \sum_i \lambda_i$, where $\lambda_i : \mathrm{End}(V_i) \to \mathbb{C}$. So $\lambda_i(\rho_{V_i}(g))$ is in $\mathcal{O}(G)$. But each $\lambda_i(\rho_{V_i}(g))$ lies in a different summand, and $\mathrm{End}(V_i)$ injects into $\mathcal{O}(G)$. Hence all the λ_i are 0, a contradiction.

Consequence 3: Let's think about the diagonal G inside $G \times G$.

$$\mathcal{O}(G)^G = \{f \in \mathcal{O}(G) \text{ such that } f(h) = f(g^{-1}hg) \text{ for all } g \in G\}$$
$$= \{\text{class functions in } \mathcal{O}(G)\}$$

Now $\mathcal{O}(G)$ is a direct sum of submodules $V^\vee \otimes V$ for V simple, and

$$(V \otimes V^\vee)^G \cong \mathrm{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{Id}$$

In $\mathcal{O}(G)$, the function is $\mathrm{Tr}(\rho_V(g)) = \chi_V(g)$. So the characters are an orthogonal basis for the class function in $\mathcal{O}(G)$.

If G is finite, we can think of G as a compact group with the discrete topology. Then $\mathcal{O}(G) = \mathbb{C}^G$ and $\dim \mathbb{C}^G = \sum \dim V^\vee \otimes V$, so

$$|G| = \sum |\dim(V)|^2$$

In this case, the space of class functions has as a basis the characters of G. So the number of conjugacy classes is equal to the number of characters.

4. An aside on conjugacy classes

Question from the floor: Is there a nice bijection between characters and conjugacy classes.

No. In particular, we cannot find a bijection such that the outer automorphisms of G respect the bijection. See `mathoverflow.net/questions/21606` and `mathoverflow.net/questions/46900` for counterexamples. (Terminology: The inner automorphisms are those induced by conjugation by a group element; the group of outer automorphisms is $\mathrm{Aut}(G)/\mathrm{Inn}(G)$. Inner automorphisms act trivially on both characters and conjugacy classes, so the outer automorphism group is the natural thing to ask about.)

Question from the floor: How does this work for S_n?

For $n \neq 6$, $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$, so the question is trivial.

For $n = 6$, there are two ways to embed S_5 in S_6. In addition to the obvious embedding, we have an embedding σ of $S_5 \cong \mathrm{PGL}_2(\mathbb{F}_5)$ into S_6. (Alternate way to describe the embedding of S_5 into S_6, not mentioned in class : The group $D := (\mathbb{Z}/4) \ltimes (\mathbb{Z}/5)$ clearly embeds in S_5, so S_5 acts on S_5/D, which has 6 elements.) So $S_6 \circlearrowleft S_6/\sigma(S_5)$ gives a nontrivial outer automorphism, which swaps the conjugacy classes of (12) and $(12)(34)(56)$. David doesn't know how it acts on the representations. See `http://arxiv.org/abs/0710.5916` for more on the outer automorphism of S_6.

5. The Full Peter-Weyl Theorem

This section is mostly for cultural enrichment.

We haven't yet shown that nontrivial finite dimensional representations exist for an arbitrary compact group. This is hard and requires analysis. The full statement of Peter-Weyl says that there are lots of finite dimensional representations of compact groups. Before we give a precise statement, motivation:

Let

$$G = S^1 = \mathbb{R}/\mathbb{Z}$$

The simple representations of G are all $\theta \mapsto e^{2\pi i k\theta}$, $k \in \mathbb{C}$. So

$$\mathcal{O}(G) = \mathrm{Span}_{\mathbb{C}} e^{2\pi i k\theta} = \{\text{finite sums } \sum a_k e^{2\pi i k\theta}\}$$

Fourier analysis gives $L^2(G) = \widehat{\oplus} \mathbb{C} \cdot e^{2\pi i k\theta}$. We want to generalize this to all compact groups.

The full statement of the Peter-Weyl Theorem is

$$L^2(G) = \widehat{\oplus} V^\vee \otimes V$$

If G is a Lie group, $f : G \to \mathbb{C}$ is smooth, then f is the absolutely convergent sum of its projections onto each $V^\vee \otimes V$.

See `http://terrytao.wordpress.com/2011/01/23/` `the-peter-weyl-theorem-and-non-abelian-fourier-analysis-on-compact-groups/` for a pretty clear proof.

Cautionary sidenote: The Peter-Weyl theorem is not quite as good as the results from Fourier analysis.

Recall the Gibbs phenomenon from Fourier analysis. The Fourier series of a sufficiently "nice" function converges away from discontinuities.

Let $G = SU(2)$. Then the elements of G can be written

$$\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

where $a^2 + b^2 + c^2 + d^2 = 1$.

Let $f : G \to \mathbb{C}$ be the function

$$f(g) = \begin{cases} 1 \text{ if } a > 0 \Leftrightarrow \mathrm{Tr}(g) > 0 \\ -1 \text{ if } a < 0 \Leftrightarrow \mathrm{Tr}(g) < 0 \end{cases}$$

Let $a = \cos(\theta)$. Then we have
$V_0 = \mathbb{C}$, $\chi_{V_0} = 1$
$V_1 = \mathbb{C}^2$, $\chi_{V_1} = e^{i\theta} + e^{-i\theta}$.
$V_k = \mathrm{Sym}^k \mathbb{C}^2$, $\chi_{V_k} = e^{ik\theta} + e^{i(k-2)\theta} + \cdots + e^{-ik\theta}$.
We would expect to have $f(\theta) = \sum c_k \chi_k(\theta)$ where $c_k = \int_{SU(2)} f(g) \chi_k(g)$.

In fact, at $g = \mathrm{Id}$, the sum does not converge, but alternates between $1 - 2/\pi$ and $1 + 2/\pi$. Here is a plot of the sum of the first 20 terms, together with the step function it is supposed to be approaching; observe that the value at 0 is $0.3619 \approx 1 - 2/\pi$. However, in an L^2 sense, the sum does converge to f; the dip at 0 get's narrower as we add more terms.
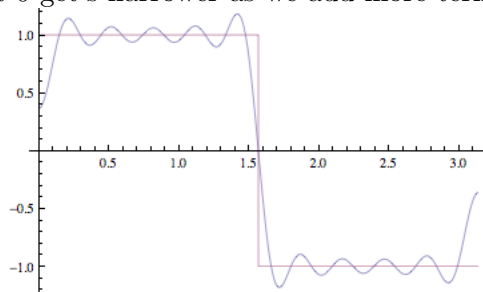
So the discontinuity at the equator gets "focused" at the North Pole, even though the f is continuous there. See `http://sbseminar.wordpress.com/2011/02/18/a-peter-weyl-counter-example/` for more on this example.