MATH 665 PROBLEM SET 1: DUE SEPT 17

See the course website for homework policy.

Problem 1 Use the mathematical software package of your choice to do these computations:

(a) Write down the 6 transition matrices relating the e, h and m bases for the degree 4 symmetric polynomials. (These should be 5×5 matrices.)

(b) Expand $(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)$ in the e-basis of Λ_4.

Problem 2 Let λ and μ be two partitions with $|\lambda| = |\mu|$. Show that $\lambda \preceq \mu$ if and only if $\mu^T \preceq \lambda^T$.

Problem 3 Let $\phi : GL_2(\mathbb{C}) \to \mathbb{C}$ be a continuous funtion which obeys $\phi(ghg^{-1}) = \phi(h)$. Show that $\phi\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right) = \phi\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$.

Problem 4 ***This problem has been modified*** In class, we described the standard inclusion $\Lambda_n \to \Lambda$ by sending $h_\lambda \mapsto h_\lambda$ for $\ell(\lambda) \leq n$. Show that this is not a map of rings.

Problem 5

(a) Show that

$$\prod_{i,j}(1+x_iy_j) = \sum_{\lambda} e_\lambda(x)m_\lambda(y).$$

(b) Define $b_{\lambda\mu} = \langle e_\lambda, h_\mu \rangle$. Show that $b_{\lambda\mu} = b_{\mu\lambda}$.

(c) Show that the involution ω preserves $\langle\ ,\ \rangle$

(d) In class, we gave an interpretation of $\langle h_\lambda, h_\mu \rangle$ in terms of matrices of nonnegative integers with given row and column sum. Give a similar interpretation of $b_{\lambda\mu}$.

Problem 6 This problem will work you through the basic properties of the power symmetric functions; they are important but which won't come up very often for us. It will provide our first (but not best) proof that $\langle\ ,\ \rangle$ is positive definite.

Define $p_k(x) = \sum x_i^k$ and define $p_\lambda(x) = \prod_i p_{\lambda_i}(x)$.

(a) Prove Newton's Identity:

$$ke_k = e_{k-1}p_1 - e_{k-2}p_2 + e_{k-3}p_3 - \cdots \pm e_1 p_{k-1} \mp p_k.$$

(b) Show that $\Lambda \otimes \mathbb{Q}$ is $\mathbb{Q}[p_1, p_2, p_3, \ldots]$.

(c) Establish a formula of the form

$$\prod_{i,j} \frac{1}{1-x_iy_j} = \sum_{\lambda} z_\lambda p_\lambda(x) p_\lambda(y)$$

for some positive rational numbers z_λ. Hint: $\frac{1}{1-w} = \exp(w + w^2/2 + w^3/3 + w^4/4 + \cdots)$.

(d) What is $\langle p_\lambda, p_\mu \rangle$?