MATH 665 PROBLEM SET 10: DUE DEC 3

See the course website for homework policy.
This problem set has two sides!

Problem 1 Let V be the standard representation of GL_i. Let $\rho : GL_{i+1} \to GL(W)$ be a representation and let $\sigma : \mathfrak{gl}_{i+1} \to \mathrm{End}(W)$ be the corresponding map of Lie algebras. Define a map $m : V \times W \to W$ by

$$m(v, w) = \sigma \begin{pmatrix} 0 & 0 & \cdots & 0 & v_1 \\ 0 & 0 & \cdots & 0 & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & v_i \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} w.$$

(a) Show that m is bilinear, so it extends to a map $\tilde{m} : V \otimes W \to W$. Show that $\tilde{m}$ is a map of GL_i representations, where GL_i acts on W by restricting the GL_{i+1} action.

(b) Decompose W into GL_i isotypic components and let $v \in V$. Suppose that the map $\tilde{m}(v, \ )$ from the λ-component to the μ-component is nonzero. Show that $\mu \supset \lambda$ and μ/λ is a single box.

(c) Let x_T be the Gelfand-Tsetlin basis of $V_\lambda(n)$. Show that $\sigma(E_{i(i+1)})x_T$ is in the span of the vectors x_U, where U ranges over vectors obtained by changing a single i of T to an $i+1$.

Problem 2 Inside the word crystal for $\mathfrak{gl}_3$, find and draw the connected components of the words 1, 12, 11, 112 and 121.

Problem 3.(a) Let $w = w_1 w_2 \cdots w_d$ be a word in the alphabet $\{1, 2, \dots, n\}$. Define $\alpha(w)$ to be the word $(n+1-w_d)(n+1-w_{d-1}) \cdots (n+1-w_1)$. For example, if $n = 3$, then $\alpha(12331312323113) = 12212123131123$. Show that $\alpha(e_k(w)) = f_{n-k}(\alpha(w))$ and $\alpha(f_k(w)) = e_{n-k}(\alpha(w))$.

(b) Let $w = w_1 w_2 \cdots w_d$ be a word in the alphabet $\{1, 2, \dots, n\}$. For $i \in \{1, 2, \dots, n\}$, define $\bar{i}$ to be the word $12 \cdots (i-1)(i+1) \cdots n$. Define $\beta(w)$ to be the concatenation $\bar{w}_d \bar{w}_{d-1} \cdots \bar{w}_1$. For example, with $n = 3$, $\beta(1213) = 12231323$. Show that $\beta(e_k(w)) = f_k(\beta(w))$ and $\beta(f_k(w)) = e_k(\beta(w))$.

(c) Obviously, $\beta(\beta(w)) \neq w$ (except for $n = 2$). However, show that the component of the word crystal containing w is isomorphic to the component of the word crystal containing $\beta(\beta(w))$.

Problem 4 This problem takes place inside the word crystal for $\mathfrak{gl}_2$. For a binary word u, let $a(u)$ be the number of ones in u, let $b(u)$ be the number of twos and let $\ell(u)$ be the length of the string containing u. (**Convention check:** $\bullet \leftrightarrows \bullet \leftrightarrows \bullet \leftrightarrows \bullet$ has length 3.)

(a) Let u and v be two binary words and let uv be their concatenation. Show that $(a(uv), b(uv), \ell(uv))$ is determined by $(a(u), b(u), \ell(u))$ and $(a(v), b(v), \ell(v))$.

(b) Let $(u_0, u_1, \dots, u_\ell)$ and $(v_0, v_1, \dots, v_m)$ be strings. Show that the set of concatenations $u_i v_j$ is closed under the crystal operators; draw the graph of how the crystal operators act on this set.

Problem 5 (The RSK correspondence) In this problem, we give a bijective proof of

$$\sum_\lambda s_\lambda(x_1, x_2, \dots, x_n) s_\lambda(y_1, y_2, \dots, y_n) = \prod_{i,j} \frac{1}{1 - x_i y_j}. \quad (*)$$

This proof isn't directly relevant to recent topics, but stay tuned.

Recall the definition of horizontal strips from the Pieri rule. We'll write $\alpha \uparrow \beta$ to mean that β/α is a horizontal strip, and $\alpha \downarrow \beta$ to mean that α/β is a horizontal strip.

(a) Fix partitions α and γ. Give a bijective proof that

$$\sum_{\alpha \uparrow \beta \downarrow \gamma} x^{|\beta/\alpha|} y^{|\beta/\gamma|} = \left(\sum_{\alpha \downarrow \beta' \uparrow \gamma} y^{|\alpha/\beta'|} x^{|\gamma/\beta'|} \right) \left(\sum_{k=0}^{\infty} (xy)^k \right).$$

(b) Show that

$$\sum_{\lambda} s_\lambda(x_1, x_2, \ldots, x_n) s_\lambda(y_1, y_2, \ldots, y_n) = \sum_{\emptyset \uparrow \beta_1 \uparrow \beta_2 \uparrow \cdots \uparrow \beta_{n-1} \uparrow \lambda \downarrow \delta_{n-1} \downarrow \cdots \downarrow \delta_2 \downarrow \delta_1 \downarrow \emptyset} x_1^{|\beta_1|} x_2^{|\beta_2/\beta_1|} \cdots x_n^{|\lambda/\beta_{n-1}|} y_n^{|\lambda/\delta_{n-1}|} \cdots y_2^{|\delta_2/\delta_1|} y_1^{|\delta_1|}.$$

(c) Prove the equality $(*)$.

(d) Describe a weight preserving bijection between ordered pairs (T, U) of SSYT with shape$(T) =$ shape(U) and rectangular arrays of nonnegative integers.

Attribution: I learned this proof from `arXiv:1210.7109`. This bijection is not hard to relate to the bijection in `arXiv:math/0703414`; the main result of that paper is that this bijection is RSK.