MATH 665 PROBLEM SET 11: DUE DEC 10
CRYSTALS AND THE PETER-WEYL THEOREM

See the course website for homework policy.

The goal of this problem set is to present a crystal analogue of $\mathbb{C}[z_{ij}] \cong \bigoplus_\lambda V_\lambda^\vee \otimes V_\lambda$.

Fix m and n. Let B be the set of nonnegative integer matrices with m rows and n columns. We will put two crystal structures on B. The first one, which we call the ***row crystal*** and denote $(\mathrm{wt}^R, e_i^R, f_i^R)$, will be a $\mathfrak{gl}_n$ crystal; the second, which we call the ***column crystal*** and denote $(\mathrm{wt}^C, e_i^C, f_i^C)$, will be a $\mathfrak{gl}_m$ crystal.

For the row crystal, we embed B into the word crystal on n letters by sending the matrix A_{ij} to

$$n^{A_{1n}}(n-1)^{A_{1(n-1)}} \cdots 2^{A_{12}} 1^{A_{11}} n^{A_{2n}}(n-1)^{A_{2(n-1)}} \cdots 2^{A_{22}} 1^{A_{21}} \cdots n^{A_{mn}}(n-1)^{A_{m(n-1)}} \cdots 2^{A_{m2}} 1^{A_{m1}}$$

For example

$$\begin{pmatrix} 1 & 3 & 5 \\ 4 & 2 & 3 \end{pmatrix} \longmapsto 3^5 2^2 1^1 3^3 2^2 1^4 = 333332213332221111$$

We'll call this the ***row word***.

Problem 1 Check that $B \sqcup \{0\}$ is closed under the operators of the word crystal. If $e_j^R(A) \neq 0$ then show that, for some index i, we have $e^R(A)_{ij} = A_{ij} + 1$, $e^R(A)_{i(j+1)} = A_{i(j+1)} - 1$, and $e^R(A)_{pq} = A_{pq}$ for $(p,q) \neq (i,j)$, $(i, j+1)$. In this case, we will say that e_j^R ***acts on*** A ***in row*** i.*

For the column crystal, we embed B into the word crystal on m letters in a similar way, but reading up the columns, with the columns ordered from left to right. For example,

$$\begin{pmatrix} 1 & 3 & 5 \\ 4 & 2 & 3 \end{pmatrix} \longmapsto 2^4 1^1 2^2 1^3 2^3 1^5 = 222212211122211111$$

We'll call this the ***column word***.

Problem 2 Consider 2×2 nonnegative integer matrices with $A_{11} + A_{12} + A_{21} + A_{22} = 2$. Draw the graph of the row and column operators acting on this set. (Hint: It has 10 vertices and 2 connected components.)

Problem 3.(a) Suppose that e_j^R acts on A in row i. Show that

$$A_{(i-1)j} < A_{i(j+1)} \quad \text{and} \quad A_{ij} \geq A_{(i+1)(j+1)}$$

(b) Describe how the $(i-1, i)$ and $(i, i+1)$ mountain ranges for the column words of A and $e_j^R(A)$ differ from each other.

(c) Show that e_j^R commutes with e_i^C and f_i^C.

A similar argument (which you need not give) shows that f_j^R commutes with e_i^C and f_i^C

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*In the row crystal, we read along rows, operators act in rows, and weights are obtained by adding up columns. It is the analogue of the right action on $m \times n$ matrices, where GL_n said both to "act on rows" and "act by column operations". I went back and forth as to whether this should be called the "row crystal" or the "column crystal"; neither terminology is standard.

Problem 4.(a) Let G be a connected component of the graph whose vertex set is V and which has edges for both the row and the column crystal operators. Show that $G \cong G_R \times G_C$ for some connected crystals G_R and G_C. (Here the product structure is that G has a vertex (u, v) for each ordered pair (u, v) of vertices in G_R and G_C respectively, and has edges $((u_1, v), (u_2, v))$ and $((u, v_1), (u, v_2))$ corresponding to edges (v_1, v_2) and (u_1, u_2) of G_R and G_C respectively.)

(b) In the above setting, show that $G_R \cong \mathrm{Crys}(\lambda)$ and $G_C \cong \mathrm{Crys}(\mu)$ for some μ.

Problem 5.(a) Let A be high weight for the row operators. Show that $A_{ij} = 0$ for $i < j$.

(b) Describe those matrices A which are high weight for both the row and column operators. (You might want to check your description against your work in problem 2.)

(c) Show that

$$B \cong \bigsqcup_{\lambda} \mathrm{Crys}(\lambda) \times \mathrm{Crys}(\lambda)$$

Problem 6.(a) Let B^R_{hi} be the high row elements for the row structure. Show that $B^R_{\mathrm{hi}} \sqcup \{0\}$ is a closed under the column operators, and forms a crystal isomorphic to $\bigsqcup_\lambda \mathrm{Crys}(\lambda)$.

(b) Let A in B with $A_{ij} = 0$ for $i < j$. Show that the following are equivalent:

(1) A is in B^R_{hi}

(2) For $m > i \geq j \geq 1$, we have

$$\sum_{r=i}^{j} A_{rj} \geq \sum_{r=i+1}^{j+1} A_{r(j+1)}$$

(3) There is a (necessarily unique) SSYT of shape $(\sum_r A_{r1}, \sum_r A_{r2}, \ldots, \sum_r A_{rm})$ containing A_{ij} copies of i in row j.

I don't suggest checking this, but the bijection between B^R_{hi} in (b) realizes the crystal isomorphism which you proved to exist by pure thought in (a).

The crystal perspective on RSK: Let $A \in B$. Under the isomorphism $B \cong \bigsqcup_\lambda \mathrm{Crys}(\lambda) \times \mathrm{Crys}(\lambda)$, the matrix A sits in some connected component $G_R \times G_C$. Let A_R be the high weight element of G_R and A_C be a high weight element of G_C. Using problem 6, we can identify A_R and A_C with SSYT T_R and T_C. The map $A \mapsto (T_R, T_C)$ is a bijection between {nonnegative integer matrices} and {ordered pairs (T_R, T_C) of SSYT of the same shape}. This is the RSK (Robinson-Schensted-Knuth) map, which can be constructed in many other ways; you saw one such way on the previous problem set. The content of T_R (respectively T_C) gives the weight of A for the column (resp. row) crystal structure and is thus equal to the row (resp. column) sums of A.

There are two important special cases of RSK. First, let $m = n$ and restrict your attention to permutation matrices. These have weight $(1, 1, \ldots, 1)$ for both crystal structures. So we get a bijection between S_n and {ordered pairs (T_R, T_C) of *standard* Young tableaux of the same shape and size n}. This is the analogue of $\mathbb{C}[S_n] \cong \bigoplus_{|\lambda|=n} Sp(\lambda) \otimes Sp(\lambda)$.

We can also impose that there be one entry in each row but not impose this on the columns. Then our matrices are naturally identified with words in $\{1, 2, \ldots, n\}^m$. We biject such words with ordered pairs (T_R, T_C) where T_R is standard but T_C is semistandard. This is the analogue of our isomorphisms $(\mathbb{C}^n)^{\otimes m} \cong \bigoplus_{|\lambda|=m} Sp(\lambda) \otimes V_\lambda(n)$. In particular, it gives us another way of seeing the isomorphism of crystals $\{1, 2, \ldots, n\}^m \cong \bigsqcup_{|\lambda|=m} \#(SYT(\lambda))\ \mathrm{Crys}(\lambda)$.

Attribution: This material seems to have been discovered at least twice: By Danilov and Koshevoy, *Arrays and the combinatorics of Young tableaux* and by van Leeuwen, *Double crystals of binary and integral matrices.*

Thank you for taking my course!