MATH 665 PROBLEM SET 3: DUE OCT 1

See the course website for homework policy. Most of these problems could also have been assigned the previous week.

Problem 1 Find a simple formula for $s_{(n-1)(n-2)\cdots 321}(x_1, x_2, \dots, x_n)$.

Problem 2 (a) Fix a positive integer d and let $\lambda = (\lambda_1, \dots, \lambda_d)$. Let $\delta(\lambda, d)$ be $s_\lambda(1, 1, \dots, 1, 0, 0, \dots)$ where there are d ones and all the other variables are 0. Show that $\delta(\lambda, d)$ is a polynomial in d. Hint: Some definitions of Schur functions make this much easier than others.

(b) Find explicit formulas for $\delta((k), d)$, $\delta(1^k, d)$, $\delta((2,1), d)$ and $\delta((2,2), d)$.

(c) A ***standard Young tableau*** is an SSYT all of whose entries are distinct. State and prove a relation between $\delta(\lambda, d)$ and the number of standard Young tableau of shape λ.

Problem 3 This problem concerns walks in a graph whose vertices are $\mathbb{Z}^2$; the steps of these walks travel in directions $(1,0)$ and $(0,1)$.

(a) Describe a way of weighting the edges of the graph, so that the weighted sum of paths from $(0,0)$ to $(k, n-k)$ is $e_k(x_1, x_2, \dots, x_n)$.

(b) Let $(\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition. Find a problem about counting paths whose answer is

$$\det \begin{pmatrix} e_{\lambda_1} & e_{\lambda_1+1} & e_{\lambda_1+2} & \cdots & e_{\lambda_1+n-1} \\ e_{\lambda_2-1} & e_{\lambda_2} & e_{\lambda_2+1} & \cdots & e_{\lambda_2+n-2} \\ e_{\lambda_3-2} & e_{\lambda_3-1} & e_{\lambda_3} & \cdots & e_{\lambda_3+n-3} \\ \vdots & & & \ddots & \\ e_{\lambda_n-n+1} & e_{\lambda_n-n+2} & e_{\lambda_n-n+3} & \cdots & e_{\lambda_n} \end{pmatrix}$$

where all symmetric polynomials are in x_1, x_2, $\dots$, x_n.

(c) Find another of counting noncrossing paths to show that this determinant is equal to $s_{\lambda^T}(x_1, x_2, \dots, x_n)$. This is called the ***dual Jacobi-Trudy indentity***.

Problem 4 The goal of this problem is to sketch another proof of $\prod(1-x_iy_j)^{-1} = \sum s_\lambda(x)s_\lambda(y)$, in terms of the ratios of alternants formula. More precisely, our goal is to show:

$$\prod_{1 \leq i,j \leq n} \frac{1}{1-x_iy_j} = \sum_{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n} \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\prod_{1\leq i<j\leq n}(x_i-x_j)} \cdot \frac{\det\left(y_i^{\lambda_j+n-j}\right)}{\prod_{1\leq i<j\leq n}(y_i-y_j)}. \quad (*)$$

(a) Prove Cauchy's determinant identity

$$\det\left(\frac{1}{u_i+v_j}\right)_{1\leq i,j\leq n} = \frac{\prod_{1\leq i<j\leq n}(u_i-u_j)\prod_{1\leq i<j\leq n}(v_i-v_j)}{\prod_{1\leq i,j\leq n}(u_i+v_j)}.$$

Hint: Let the determinant be $p(u,v)/\prod(u_i+v_j)$. What can you say about p?

(b) Make a change of variables in part (a) to show:

$$\det\left(\frac{1}{1-x_iy_j}\right)_{1\leq i,j\leq n} = \frac{\prod_{1\leq i<j\leq n}(x_i-x_j)\prod_{1\leq i<j\leq n}(y_i-y_j)}{\prod_{1\leq i,j\leq n}(1-x_iy_j)}.$$

(c) Show that

$$\det\left(\frac{1}{1-x_iy_j}\right)_{1\leq i,j\leq n} = \sum_{k_1,k_2,\dots,k_n\geq 0} \det\left(x_i^{k_j}y_j^{k_j}\right).$$

(d) Prove equation $(*)$