MATH 665 PROBLEM SET 4: DUE OCT 8

See the course website for homework policy.

Problem 1 Let $G = \mathbb{Z}$, let V be a two dimensional vector space, and let ρ be the representation $G \to GL_2(V)$ by $k \mapsto \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$.

(a) Show that V contains a one dimensional subrepresentation U.

(b) Show that there does not exist a second subrepresentation W with $V = U \oplus W$.

Problem 2 Let $\rho : G \to GL(V)$ be a representation with character χ. Let ψ be the character of $\bigwedge^2 V$. Prove the identity:

$$\psi(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right).$$

Remark: Recall the power symmetric functions from problem set 1. Let f_k be the polynomial such that $e_k = f_k(p_1, p_2, p_3, \ldots)$ and let ψ_k be the character of $\bigwedge^k V$. One can generalize the argument of this problem to show that $\psi_k(g) = f_k(\chi(g), \chi(g^2), \chi(g^3), \ldots)$.

Problem 3 Let G be a finite group, and let $W_1, W_2, \ldots, W_n$ be a sequence of nonzero representations of G. Suppose that

$$\mathbb{C}^G \cong \bigoplus W_i \otimes W_i^\vee$$

as $G \times G$ representations. Show that W_i's are the irreducible representations of G, each listed once.

Problem 4 Let A be an abelian group and let $\rho : A \to GL(V)$ be a simple G-representation.

(a) Show that $\rho(a)$ is a scalar multiple of the identity matrix for all $a \in A$.

(b) Show that V is one dimensional.

Problem 5 Let G be a finite group and X a finite set with an action of G. Let $\mathbb{C}X$ be the vectors space of functions $X \to \mathbb{C}$, with the obvious action of G. Let χ be the character of $\mathbb{C}X$.

(a) Give a simple description of $\chi(g)$.

(b) Show by a direct argument $\dim(\mathbb{C}X)^G$ is the number of orbits of G acting on X.

(c) Compare (b) with the formula $\dim V^G = \int_G \chi(g) dg$ proved in class. What standard group theory result have you reproved?

(d) Show that $\frac{1}{|G|}\sum_{g \in G} \chi(g)^2$ is the number of orbits of G acting on $X \times X$.

(e) Let G act transitively on X, and let $U \subset \mathbb{C}X$ be the space of functions $f : X \to \mathbb{C}$ with $\sum_{x \in X} f(x) = 0$. Show that U is irreducible if and only if G acts transitively on the set of ordered pairs $\{(x, y) \in X^2,\ x \neq y\}$.

Problem 6 Let G be a compact group and V an irreducible representation of G. Let A be any element of $\mathrm{End}(V)$. Show that

$$\int_G \rho(g) \cdot A \cdot \rho(g)^{-1} dg \;=\; \frac{Tr(A)}{\dim V}\mathrm{Id}.$$

Hint: First show that the left hand side is a scalar multiple of the identity.