MATH 665 PROBLEM SET 5: DUE OCT 17 (WEDNESDAY)

See the course website for homework policy.

Problem 1 Let ρ be a representation of GL_n, with character given by the symmetric function $f(x_1, x_2, \dots, x_n)$. Let the restriction of ρ to $GL_k \times GL_{n-k}$ have character $g(y_1, \dots, y_k, z_1, z_2, \dots, z_{n-k})$, a function symmetric in the y's and the z's. What is the relationship between f and g?

Problem 2 Let V denote the standard n dimensional representation of GL_n. Compute the character of $\bigwedge^2(\bigwedge^2 V)$ as a symmetric polynomial.

Problem 3

Let $\rho : GL_n(\mathbb{C}) \to GL(V)$ be a polynomial representation of $GL_n(\mathbb{C})$. Write $\rho(g) = \sum_k \rho_k(g)$ where the entries of $\rho_k(g)$ are degree k polynomials in the entries of g.

(a) Show that $\rho_k(g)\rho_\ell(h) = 0$ for $k \neq \ell$.

(b) Show that $\rho_k(g)\rho_k(h) = \rho_k(gh)$.

(c) Show that V can be written as $\bigoplus V_k$, where $GL_n(\mathbb{C})$ acts on V_k by a polynomial representation whose entries are homogenous polynomials of degree k.

Problem 4 Let G be a compact group and $\rho : G \to GL(V)$ an irrep. Choose a basis for V such that $\rho(g)$ is unitary. The aim of this problem is to prove the formula

$$\int_G \rho_{ij}(g)\overline{\rho_{k\ell}(g)}dg = \frac{1}{\dim V}\delta_{ik}\delta_{j\ell}. \quad (*)$$

(a) Let $\sigma : G \to GL(\mathrm{End}(V))$ be the representation $\sigma(g)(A) = \rho(g)A\rho(g)^{-1}$. Construct a linear map $\lambda : \mathrm{End}(\mathrm{End}(V)) \to \mathbb{C}$ so that

$$\int_G \rho_{ij}(g)\overline{\rho_{k\ell}(g)}dg = \int_G \lambda(\sigma(g))dg.$$

(b) Compute $\int_G \sigma(g)dg$, so your answer should be an element in $\mathrm{End}(\mathrm{End}(V))$. Hint: You already did this on problem set 4.

(c) Prove formula $(*)$.

Problem 5

Let G be the unitary group $U(n)$ and let $\mathcal{O}(G)$ be the ring of matrix coefficients. In class, we proved that $\mathcal{O}(G)$ was generated as a $\mathbb{C}$ algebra by the matrix entries z_{ij} and their complex conjugates. This problem presents another proof of that fact.

(a) Show that, if $X \subset Y$ are two representations of G, then the image of $\mathrm{End}(X)^\vee$ in $\mathcal{O}(G)$ is contained in the image of $\mathrm{End}(Y)^\vee$.

Let V be the standard n-dimensional representation of G; let $W = V \oplus V^\vee \oplus \mathbb{C}$.

(b) Show that, for any N, the elements of $\mathrm{End}(W^{\otimes N})^\vee$ in $\mathcal{O}(G)$ are polynomials in z_{ij} and $\overline{z_{ij}}$.

(c) Let χ be the character of W. Show that $\chi(\mathrm{Id}) = 2n + 1$ and $|\chi(g)| < 2n + 1$ for all other $g \in G$.

(d) Let U be a (nonzero) irrep of G, with character ψ. Show that, for N sufficiently large, $\int_G \chi(g)^N\overline{\psi}(g)dg > 0$.

(e) Finish the proof.

One advantage of this proof is that it can be adapted to show that, for K any closed subgroup of $U(n)$, the ring $\mathcal{O}(K)$ is generated by z_{ij} and $\overline{z_{ij}}$.