MATH 665 PROBLEM SET 6: DUE OCT 22

See the course website for homework policy.

Problem 1 (The Weyl integration formula) Let $A(x_1, x_2, \ldots, x_n)$ be the Vandermonde determinant $\det(x_i^{j-1})_{1 \le i,j \le n}$. Set $s_{\lambda - N}(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n)(x_1 \cdots x_n)^{-N}$.

(a) Define $W(\theta_1, \ldots, \theta_n)$ to be $A(e^{i\theta_1}, \ldots, e^{i\theta_n})A(e^{-i\theta_1}, \ldots, e^{-i\theta_n})$. Show that

$$W(\theta_1, \ldots, \theta_n) = 2^{n(n-1)} \prod_{i<j} \sin^2\left(\frac{\theta_i - \theta_j}{2}\right).$$

(b) Show that

$$\int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} s_{\lambda-N}(e^{i\theta_1}, \ldots, e^{i\theta_n}) W(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n = \begin{cases} (2\pi)^n n! & \text{if } \lambda = (N, N, \ldots, N) \\ 0 & \text{otherwise} \end{cases}.$$

(c) Let V be a representation of GL_n and χ_V its character. Show that

$$\frac{1}{(2\pi)^n n!} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} \chi_V(e^{i\theta_1}, \ldots, e^{i\theta_n}) W(\theta_1, \ldots, \theta_n) d\theta_1 \cdots d\theta_n = \dim V^G.$$

(d) Let F be any class function in $\mathcal{O}(U(n))$. Show that

$$\frac{1}{(2\pi)^n n!} \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} F(e^{i\theta_1}, \ldots, e^{i\theta_n}) W(\theta_1, \ldots, \theta_n) d\theta_1 \ldots d\theta_n = \int_{U(n)} F(g) dg.$$

This is the ***Weyl integration formula***.

(e) (Optional) Show that the same formula holds for F any continuous class function on $U(n)$.

Interpretation: We can kill the factor of $n!$ by integrating over the smaller domain $D := \{0 \le \theta_1 \le \theta_2 \le \cdots \le \theta_n < 2\pi\}$. D meets every conjugacy class of $U(n)$ exactly once. We have

$$\frac{1}{(2\pi)^n} \int_D F(e^{i\theta}) W(\theta) d\theta = \int_{U(n)} F(g) dg.$$

One can think of this as saying that the conjugacy class of $\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ has volume $W(\theta)/(2\pi)^n$. Using an invariant metric on $U(n)$ to define volumes of submanifolds, this can be made precise.

Problem 2 (a) Find the coefficient of $wxyz$ in $s_{31}(w, x, y, z)$ and $s_{22}(w, x, y, z)$.

(b) Find a basis for the part of $\mathbb{S}_{31}\mathbb{C}^4$ which has weight $wxyz$ for the action of $\mathrm{diag}(w, x, y, z)$.

(c) Find a basis for the part of $\mathbb{S}_{22}\mathbb{C}^4$ which has weight $wxyz$ for the action of $\mathrm{diag}(w, x, y, z)$.

Problem 3 (The first fundamental theorem of invariant theory, preview) Let V be a finite dimensional vector space with $\dim V \ge n$. In this problem, you will show that the ring $\mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n})$ is isomorphic to $\mathbb{C}[S_n]$. The first fundamental theorem of invariant theory says that, even when $\dim V < n$, the natural map $\mathbb{C}[S_n] \longrightarrow \mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n})$ is surjective. We'll prove this in November.

(a) Use some result we have already proved to show that $\dim \mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n})$ is $n!$.

(b) Let $\alpha \in \mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n})$. Let $e_1, \ldots, e_n$ be linearly independent elements of V. Show that $\alpha(e_1 \otimes e_2 \otimes \cdots \otimes e_n)$ is in the span of $e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)}$, for σ running through S_n.

(c) In the above notation, let $\alpha(e_1, e_2, \ldots, e_n) = \sum_{\sigma \in S_n} c_\sigma e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}$. Show that, for any linearly independent $f_1, f_2, \ldots, f_n$ in V, we have

$$\alpha(f_1, f_2, \ldots, f_n) = \sum_{\sigma \in S_n} c_\sigma f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}.$$

(d) Show that the above formula holds for all $f_1, \ldots, f_n \in V$.

(e) Explain why $\mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n}) \cong \mathbb{C}[S_n]$.