MATH 665 PROBLEM SET 7: DUE NOV 5 (TWO WEEK PROBLEM SET)

See the course website for homework policy.

Note that this problem set has two pages. It's not as long as it looks!

Problem 1 Let λ and μ be partitions of n, and let[1] $\dim V \geq |\lambda| = |\mu|$. We proved many problem sets ago that $\langle e_\lambda, h_\mu \rangle$ is the number of $(0,1)$ matrices with row sum λ and column sum μ. Let

$$H = \bigotimes_k \mathrm{Sym}^{\lambda_k}(V) \text{ and } E = \bigotimes_k \bigwedge^{\mu_k^T}(V).$$

So $\dim \mathrm{Hom}_{GL(V)}(E, H)$ is the number of such $(0,1)$ matrices. In this problem, we will construct a basis for $\dim \mathrm{Hom}_{GL(V)}(E, H)$ indexed by these matrices.

For a partition γ of n, let S_γ be the subgroup $S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_k}$ of S_n, where the first factor permutes $\{1, 2, \ldots, \gamma_1\}$, the second factor permutes $\{\gamma_1+1, \gamma_1+2, \ldots, \gamma_1+\gamma_2\}$ and so forth. Define

$$a_\gamma = \frac{1}{|S_\gamma|} \sum_{w \in S_\gamma} w \text{ and } b_\gamma = \frac{1}{|S_{\gamma^T}|} \sum_{w \in S_{\gamma^T}} (-1)^w w.$$

We let the group S_n act on $V^{\otimes n}$ by permuting the factors. We extend this linearly to make $V^{\otimes n}$ into a $\mathbb{C}[S_n]$-module, so a_λ and b_μ act on $V^{\otimes n}$.

(a) Show that $a_\lambda^2 = a_\lambda$, $b_\mu^2 = b_\mu$, $a_\lambda V^{\otimes n} = H$ and $b_\mu V^{\otimes n} = E$.

(b) Recall from problem set 6 that $\mathrm{Hom}_{GL(V)}(V^{\otimes n}, V^{\otimes n}) = \mathbb{C}[S_n]$. Using this, give an isomorphism $\mathrm{Hom}_{GL(V)}(E, H) \cong a_\lambda \mathbb{C}[S_n] b_\mu$.

(c) For a permutation w in S_n, give a criterion for when $a_\lambda w b_\mu = 0$.

(d) For two permutations u and $v \in S_n$ with $a_\lambda u b_\mu$ and $a_\lambda v b_\mu$ nonzero, give a criterion for when $a_\lambda u b_\mu = \pm a_\lambda v b_\mu$.

Let N be the set of w in S_n such that $a_\lambda w b_\mu \neq 0$. Define an equivalence relation $\sim$ on N by $u \sim v$ if $a_\lambda u b_\mu = \pm a_\lambda v b_\mu$.

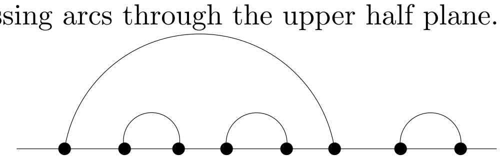
(e) Give a bijection between $N/\sim$ and $(0,1)$ matrices with row sum λ and column sum μ.

(f) Show that $N/\sim$ is a basis for $\mathrm{Hom}_{GL(V)}(E, H)$.

Problem 2 The number of standard young Tableaux of shape (k,k) is the Catalan number $\frac{1}{k+1}\binom{2k}{k}$. This problem explores different ways to think about this. Hints for this problem are encoded using ROT13, as I think it is much more fun to solve it without the hints.

(a) A ballot sequence of length $2k$ is a sequence $a_0, a_1, a_2, \ldots, a_{2k}$ with $a_0 = a_{2k} = 0$, $a_{j+1} = a_j \pm 1$ and $a_j \geq 0$. It is well known that the number of ballot sequences is equal to the Catalan number. Give a bijection between SYT of shape (k,k) and ballot sequences. Hint: Ybbx ng gur fhofrg bs gur gnoyrnh bpphcvrq ol gur ahzoref 1 guebhtu j.

(b) A noncrossing matching of $2k$ points is a partition of $\{1, 2, \ldots, 2k\}$ into k pairs, so that we can join the pairs by noncrossing arcs through the upper half plane.



A noncrossing matching of 8 points ($k = 4$)

There are Catalan many noncrossing matchings. Give a bijection between SYT of shape (k,k) and noncrossing matchings. Hint: Pbafvqre gur yrsg unaq raqf bs gur zngpuvat nepf.

(c) Define a ***two-dimensional ballot sequence*** (this term is not standard) to be a sequence of vectors $a_0, a_1, \ldots, a_{3k}$ in $\mathbb{Z}^2$ with $a_0 = a_{3k} = (0,0)$, $a_{j+1} - a_j$ equal to one of $(1,0)$, $(-1,1)$ or $(0,-1)$, and with $a_j \in \mathbb{Z}^2_{\geq 0}$ for all j. Give a bijection between two-dimensional ballot sequences and standard Young tableaux of shape (k,k,k).

[1] In fact, the main result of this problem is true when $\dim V \geq \max(\ell(\lambda), \ell(\mu^T))$ which is a weaker condition.

Problem 3 (The Gelfand-Tsetlin basis) Let λ and μ be two partitions. Recall that we say λ/μ is a vertical strip if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$. See the notes for September 24 for more. We use the notation $V_\lambda(n)$ for the simple representation of GL_n indexed by λ.

(a) Let λ be a partition with $\ell(\lambda) \leq n$. Show that

$$V_\lambda(n)|_{GL_{n-1}} \cong \bigoplus_{\substack{\lambda/\mu \text{ a vertical strip} \\ \ell(\mu) \leq n-1}} V_\mu(n-1).$$

We define a basis $v_1, v_2, \ldots, v_N$ to be a ***Gelfand-Tsetlin basis*** if, for every $k \leq n$, we can partition $\{1, 2, \ldots, N\}$ into a disjoint union $S_1^k \sqcup S_2^k \sqcup \cdots \sqcup S_M^k$ such that, for each S_i^k, the subspace $\mathrm{Span}_{s \in S_i^k}(v_s)$ is a simple GL_k subrep of $V_\lambda|_{GL_k}$.

(b) Find a Gelfand-Tsetlin basis of $V_{21}(3)$. Compute the transition matrix between this basis and the semi-standard basis.

(c) Prove that every $V_\lambda(n)$ has a Gelfand-Tsetlin basis. Describe a bijection between your basis and SSYT of shape λ with content $\leq n$.

(d) Show that, if v_i and w_i are two Gelfand-Tsetlin bases of $V_\lambda(n)$, then, after reordering the w_i, we have $w_i = c_i v_i$ for some scalars c_i. In other words, Gelfand-Tsetlin bases are unique up to reordering and rescaling.

Language note: The first character in Tsetlin's name is the Cyrillic letter ц, which is variously represented by C, Cz, Ts, Tz or Z depending on which transliteration system you use. I don't speak Russian, but I think that the sound you are looking for is the middle consonant of "matzah".