MATH 665 PROBLEM SET 8: DUE NOV 12

See the course website for homework policy.

Problem 1(a) Write down the (semi)-standard bases of $Sp(\lambda)$ for each partition λ of 4.

(b) Write down the matrices for (12), (23) and (34) acting in these bases. (This is a good task for a computer algebra system.)

Problem 2(a) Show that $c^{\nu}_{\lambda\mu} = c^{\nu^T}_{\lambda^T \mu^T}$, using symmetric polynomials.

(b) Provide an isomorphism between $\mathrm{Hom}_{GL(V)}(V_\lambda \otimes V_\mu, V_\nu)$ and $\mathrm{Hom}_{GL(V)}(V_{\lambda^T} \otimes V_{\mu^T}, V_{\nu^T})$ by identifying both vector spaces with something involving the symmetric group.

Problem 3 For this problem, fix a positive integer n. Let λ be a partition with $\ell(\lambda) \le n$; we pad λ with 0's if necessary to define λ_i for $1 \le i \le n$. Let $L \ge \lambda_1$. Define $\lambda^L = (L - \lambda_n, L - \lambda_{n-1}, \ldots, L - \lambda_1)$.

(a) What is the relation between $V_\lambda(n)^\vee$ and $V_{\lambda^L}(n)$?

(b) Let λ, μ and ν be partitions with at most n parts and with $|\nu| = |\lambda| + |\mu|$. Let $L \ge \nu_1$. Let R be the partition L^n, with L repeated n times. Give an isomorphism

$$\mathrm{Hom}_{GL_n}(V_\lambda(n) \otimes V_\mu(n), V_\nu(n)) \cong \mathrm{Hom}_{GL_n}(V_\lambda(n) \otimes V_\mu(n) \otimes V_{\nu^L}(n), V_R(n)).$$

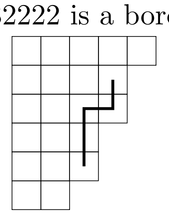
(c) Show that $c^{\nu}_{\lambda\mu} = c^{R}_{\lambda\mu\nu^L}$.

Problem 4 Let λ be a partition and let x_T be the Gelfand-Tsetlin basis of V_λ, indexed by semistandard tabelaux T of shape λ. Let ϕ map $GL_2 \to GL_n$ by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & a & b & & \\ & & c & d & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where GL_2 lands in positions i and $i+1$. Let g be an element of GL_2 and T a SSYT of shape λ. Let x_U appear with nonzero coefficient in $\phi(g) \cdot x_T$. Show that T and U differ only by changing i's to $(i+1)$'s and *vice versa*.

Problem 5 (Frobenius' Character Formula) We define μ/λ to be a border strip if the squares of μ/λ can be travelled in a single path moving up and to the right one square at a time. We define $h(\mu/\lambda)$ to be the number of vertical steps (one less than the number of rows it occupies). In the figure below, we see that 544332/532222 is a border strip of size 5 and height 3.



(a) Prove that

$$p_r s_\lambda = \sum_{\mu/\lambda \text{ a border strip}} (-1)^{h(\mu/\lambda)} s_\mu.$$

Hint: Copy the proof of Pieri's rule for computing $e_r s_\lambda$.

(b) Fix partitions ρ and μ with $|\mu| = |\rho|$. Let $C(\rho, \mu)$ be the set of all chains $\emptyset = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\ell = \mu$ such that λ_i/λ_{i-1} is a border strip of size ρ_i. Show that

$$\langle p_\rho, s_\mu \rangle = \sum_{(\lambda_\bullet) \in C(\rho,\mu)} \prod_{i=1}^{\ell} (-1)^{h(\lambda_i/\lambda_{i-1})}.$$