MATH 665 PROBLEM SET 9: DUE NOV 19

See the course website for homework policy.

Problem 1.(a) Consider the semistandard basis element of $V_{22}(4)$ corresponding to $\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}$. Compute the images of this element, in the semistandard basis, for the 16 basis elements of $\mathfrak{gl}_4$. (Hint: 6 of them will be nonzero.)

(b) Find a semistandard basis element in $V_{22}(4)$ whose image under E_{21} is nonzero and is not $\pm$ a semistandard basis element.

Problem 2 Let $\mathbb{G}_a$ be the additive group of the complex numbers. Let $\rho : \mathbb{G}_a \to GL_n(\mathbb{C})$ be a polynomial representation, meaning that the entries of $\rho(t)$ are polynomials in t. Show that $\rho(t) = \exp(Nt)$ for some nilpotent matrix N.

Problem 3 (The Casimir element) Define the element Ω, in the universal enveloping algebra of $\mathfrak{gl}_n$, to be $\sum_{1 \leq i,j \leq n} E_{ij} E_{ji}$.

(a) Show that Ω commutes with every element of $\mathfrak{gl}_n$.

(b) For $n = 2$, compute the action of Ω on $V_{ab}(2)$.

Problem 4 (The Kostant partition function) Let $\beta \in \mathbb{Z}^n$ with $\sum \beta_i = 0$. Let $N\rho = ((n-1) \cdot N, (n-2) \cdot N, \ldots, 2N, N, 0)$.

(a) Show that, for N sufficiently large, the sequence $K_{N\rho,\ N\rho+\beta}$ becomes a constant $K(\beta)$ independent of N. Here $K_{\lambda\mu}$ is the number of SSYT of shape λ and content μ.

(b) Show that $K(\beta)$ is the number of ways to write $\beta = \sum_{1 \leq i < j \leq n} a_{ij}(e_j - e_i)$, with a_{ij} an array of $\binom{n}{2}$ nonnegative integers.

Problem 5 Let $\mathfrak{gl}_2$ act on the vector space $\mathbb{C}[x]$ by

$$E_{11} \mapsto -x\frac{\partial}{\partial x} \quad E_{12} \mapsto \frac{\partial}{\partial x} \quad E_{21} \mapsto -x^2\frac{\partial}{\partial x} \quad E_{22} \mapsto x\frac{\partial}{\partial x}.$$

In this problem, we will demonstrate some of the difficulties that occur when using matrix exponentials in infinite dimensional vector spaces.

(a) Verify that this gives an action of $\mathfrak{gl}_2$ on the (infinite dimensional) vector space $\mathbb{C}[x]$.

(b) Let $f(x) \in \mathbb{C}[x]$. Show that the infinite sum defining $\exp(tE_{12})f(x)$ is convergent in $\mathbb{C}[x]$, and give a simple formula for its value.

(c) Show that the infinite sum defining $\exp(tE_{21})x$ is convergent in the ring of formal power series $\mathbb{C}[[x]]$, but not in $\mathbb{C}[x]$ (for $t \neq 0$), and evaluate that sum.

(d) Compute $\exp(tE_{12})\exp(tE_{21})x$ (this can be done either directly, or using the above results). Writing $\exp(tE_{12})\exp(tE_{21})x = a_0(t) + a_1(t)x + a_2(t)x^2 + \cdots$, find a simple formula for $a_0(t)$. You should obtain that your formula has a pole at $t = 1$.

Remark: I was originally planning to have you show that $\exp\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right) x$ blows up at $\theta = \pi/2$, but the computations were more challenging. Note that $\exp\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, so this shows that the issue can occur even when the exponential lies entirely within the unitary group.

Problem 6 This problem explores some relations between the representation theory of GL_2 and the combinatorial notion of unimodality.

(a) Let $f(x,y) = \sum f_k x^k y^{n-k}$ be a symmetric polynomial in x and y, homogeneous of degree n, with integer coefficients. Show that f is the character of a polynomial GL_2 representation if and only if $f_0 \leq f_1 \leq \cdots f_{\lfloor k/2 \rfloor} = f_{\lceil k/2 \rceil} \geq \cdots \geq f_{n-1} \geq f_n$.

(b) Let V be the standard two dimensional representation of GL_2 and apply part (a) to $V^{\otimes k}$. What standard result do you deduce?

(c) Now let W be $\mathrm{Sym}^n(V)$ and apply part (a) to $W^{\otimes k}$. What result do you deduce? To the best of my knowledge[1], there is no known direct proof of this inequality.

[1]Bruce Sagan would like a direct proof of the inequality you deduce by applying these methods to $\bigwedge^k W$, and he is certain there is none in the literature. I worked for a bit on $W^{\otimes k}$ as a warm up, but I didn't do a literature search.