LECTURE 3: COMPLETE HOMOGENOUS SYMMETRIC FUNCTIONS

SCRIBE: YI SU

Define the ***complete homogenous symmetric functions***

$$h_k = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k} x_{i_1} x_{i_2} \ldots x_{i_k}$$

Example 1.

$$h_2 = \sum_{1 \le i \le j} x_i x_j = \sum_{i \ge 1} x_i^2 + \sum_{1 \le i < j} x_i x_j = m_2 + m_{11}$$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, define

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_r}$$

Example 2.

$$\begin{aligned} h_{21} &= h_2 h_1 = (\sum_{i \ge 1} x_i^2 + \sum_{1 \le i < j} x_i x_j)(\sum_{k \ge 1} x_k) \\ &= \sum_{i \ge 1} x_i^3 + 2 \sum_{i \ne j} x_i^2 x_j + 3 \sum_{1 \le i < j < k} x_i x_j x_k \\ &= m_3 + 2m_{21} + 3m_{111} \end{aligned}$$

For future reference,

$$\begin{aligned} h_3 &= m_3 + m_{21} + m_{111} \\ h_{21} &= m_3 + 2m_{21} + 3m_{111} \\ h_{111} &= m_3 + 3m_{21} + 6m_{111} \end{aligned}$$

So we have a transformation matrix between h's and m's. For example, in degree 3, we just computed that the matrix is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$$

We have two main results today:

Theorem 3. *The h's form a basis of Λ*

Theorem 4. *The transformation matrix between h's and m's is symmetric.*

1. The h's are a basis

Since the h polynomials are the monomials in h_1, h_2, h_3, $\ldots$, the claim is that $\Lambda = \mathbb{Z}[h_1, h_2, h_3, \ldots]$.

We first show that the natural ring map $\mathbb{Z}[h_1, h_2, \ldots] \to \Lambda$ is surjective. Indeed it is enough to show every e_k is a polynomial in h's. Consider two identities

$$\prod_{i \ge 1} (1 + x_i t) = \sum_{k \ge 0} e_k t^k$$

$$\prod_{i\geq 1}\frac{1}{1-x_it}=\sum_{k\geq 0}h_kt^k$$

Hence,

$$1=\prod_{i\geq 0}(1-x_it)\prod_{i\geq 0}\frac{1}{(1-x_it)}=(\sum_{k\geq 0}(-1)^ke_kt^k)(\sum_{k\geq 0}h_kt^k)$$

Comparing the coefficients on both sides, one gets the relation:

$$h_k-e_1h_{k-1}+e_2h_{k-2}-e_3h_{k-3}+\ldots+(-1)^ke_k=0 \quad (*)$$

The dimension of degree d parts of $\mathbb{Z}[h_1,h_2,\ldots]$ and Λ are equal, which proves the injectivity. $\square$

So $\mathbb{Z}[h_1,h_2\ldots]\cong\mathbb{Z}[e_1,e_2\ldots]\cong\Lambda$.

1.1. **What about Λ_n?** We have $\Lambda_n\cong\mathbb{Z}[e_1,e_2,\ldots]/I$ where I is the ideal generated by $e_i=0$ for $i\geq n+1$. If one thinks $\Lambda\cong\mathbb{Z}[h_1,h_2\ldots]$, then $\Lambda_n\cong\mathbb{Z}[h_1,h_2\ldots]/J$, where J is the coefficient of t^k in the power series expansion of $1/(\sum_{i\geq 0}h_it^i)$ in t for $k\geq n+1$. The above proof can also show that $\Lambda_n=\mathbb{Z}[h_1,h_2,\ldots,h_n]$ so $\{h_\lambda : l(\lambda^T)\leq n\}$ is a basis for Λ_n. (In the lecture on Wednesday, we will show that $\{h_\lambda : l(\lambda\leq n)\}$ is also a basis for Λ_n).

1.2. **The map ω. (Section largely added by editor).** Noticing the symmetric between e and h, define a map $\omega:\Lambda\to\Lambda$ by $h_k\mapsto e_k$ and thus $h_\lambda\mapsto e_\lambda$. By applying ω to the equation (*), one can get $\omega(e_k)=h_k$, and hence $\omega(e_\lambda)=h_\lambda$.

From our perspective, ω is pretty mysterious. There are lots of applications of the ring of symmetric functions. In some of those other applications, ω is more motivated. For example, if you use Λ to study the cohomology of the Grassmannian $G(d,n)$, then ω is the isomorphism $G(d,n)\cong G(n-d,n)$ which sends a d-plane to ints orthogonal complement. If you use Λ to study the representation theory of S_n, then ω tensors with the sign representation. There is not a similarly elegant answer for GL_n representation theory.

Let's see what an answer would look like. First of all, what is the representation theory meaning of e_k and h_k? Remember that we go from a representation of GL_n to a symmetric polynomial by taking the trace of the action of a diagonal matrix. Let V be the standard n-dimensional representation of GL_n. In V, a diagonal matrix acts by itself, and thus has trace $x_1+x_2+\cdots+x_n=e_1=h_1$. Let's look at $\bigwedge^k V$. If V has basis e_1, e_2, $\ldots$, e_n, then a basis for $\bigwedge^k V$ is the $\binom{n}{k}$ elements $e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}$. A diagonal matrix of GL_n acts diagonally in this basis, with diagonal elements the products $e_{i_1}e_{i_2}\cdots e_{i_k}$, and hence with trace e_k. Similarly, Sym^kV will correspond to h_k.

So we want an operation which switches $\bigwedge^k V$ and Sym^kV.

Such a thing occurs in physics and is called the "boson-fermion correspondence"; I don't know much about it.

Such a thing also occurs in the theory of super-groups where, if V is the standard representation of GL_{-n}, then $\bigwedge^k V$ computed in the category of super-vector spaces is what we would normally call Sym^kV. I also don't know much about this, and we won't talk about it.

In a month, we will talk about Schur-Weyl duality, which is about the relation between GL_n and S_m representation theory. Since ω has a simple meaning on the symmetric group side, we can try to use that to extract an interpretation for ω on the GL side; let's remember to think about that.

Question from the floor: Are you saying there is no functor $GL-\text{rep} \to GL-\text{rep}$ which realizes ω? **Answer:** No, I am not willing to make such a specific claim. I am saying it is not any familiar or elegant operation.

2. The matrix is symmetric

Define $A_{\lambda\mu} =$ coefficient of m_μ in h_λ.

Theorem 5. $A_{\lambda\mu} = A_{\mu\lambda}$.

Proof. To get output $x_1^{\mu_1} x_2^{\mu_2} \dots x_r^{\mu_r}$ the h_{λ_j} term must contribute $x^{\alpha_j} = \prod_k x_k^{\alpha_j^{(k)}}$, where α_j's are vectors in $\mathbb{Z}_{\geq 0}$ with $\sum_j \alpha_j = \mu$ and $\sum_k \alpha_j^{(k)} = \lambda_j$ for all j. Thus,

$$A_{\lambda\mu} = \# \left\{ (\alpha_1, \dots, \alpha_r) | \sum_k \alpha_j^{(k)} = \lambda_j, \ \sum_j \alpha_j = \mu \right\}$$

Which is equivalent to say

$$A_{\lambda\mu} = \text{number of nonnegative integer matrices with row sum } \mu \text{ and column sum } \lambda$$

Clearly, nonnegative integer matrices with row sum μ and column sum λ is in bijection with those with row sum λ and column sum μ. Therefore, $A_{\lambda\mu} = A_{\mu\lambda}$. $\square$

Example 6. $\Lambda = (2, 1), \mu = (1, 1, 1)$*, all possible such matrices are*

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

This shows that the coefficient of m_{111} *in* h_{21} *should be* 3*.*

We can write this proof using generating function identities. By definition, we have

$$\sum_{\lambda,\mu} A_{\lambda\mu} m_\lambda(x) m_\mu(y) = \sum_\lambda m_\lambda(x) h_\lambda(y)$$

and the above proof showed that

$$\sum_{\lambda,\mu} A_{\lambda\mu} m_\lambda(x) m_\mu(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}.$$

The right hand side is sometimes known as the ***Cauchy product***. Similarly,

$$\sum_{\mu,\lambda} A_{\mu\lambda} m_\lambda(x) m_\mu(y) = \sum_\mu h_\mu(x) m_\mu(y) = \prod_{i,j \geq 1} \frac{1}{1 - y_i x_j}$$

The right hand sides are equal, so so are the left hand sides, showing $A_{\lambda\mu} = A_{\mu\lambda}$.

Let's explain the identity $\sum_{\lambda,\mu} A_{\lambda\mu} m_\lambda(x) m_\mu(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$ more slowly. Expand the geometric series in the Cauchy product

$$\begin{aligned} \prod_{i,j \geq 1} \tfrac{1}{1 - x_i y_j} &= (1 + x_1 y_1 + x_1^2 y_1^2 + \dots)(1 + x_1 y_2 + x_1^2 y_2^2 + \dots) \cdots \\ &\quad (1 + x_2 y_1 + x_2^2 y_1^2 + \dots)(1 + x_2 y_2 + x_2^2 y_2^2 + \dots) \cdots \\ &\quad \cdots \end{aligned}$$

The coefficient of $x_1^7 x_2^3 y_1^5 y_2^5$ is the number of ways in which we pick $(x_i y_j)^k$ such that the powers of x_i's is $(7,3)$, and powers of y_j's is $(5,5)$. This is in bijection with the integer matrices with nonnegative entries with row sum $(7,3)$ and column sum $(5,5)$. For example $(x_1y_1)^5(x_1y_2)^2(x_2y_2)^3$ corresponds to the matrix $\left(\begin{smallmatrix} 5 & 2 \\ 0 & 3 \end{smallmatrix}\right)$.

2.1. **The Hall inner product.** The symmetry of the matrix $A_{\lambda\mu}$ can be used in the following way: Define a bilinear product $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ is the Kronecker delta, and then $\langle h_\lambda, h_\mu \rangle = \langle h_\lambda, \sum_\nu A_{\lambda\nu} m_\nu \rangle = A_{\lambda\mu}$. Since $A_{\lambda\mu} = A_{\mu\lambda}$, this bilinear form is symmetric.

Here is a useful fact about this bilinear form, which we'll prove next time.

Proposition 7. *Let $\{f_\lambda\}$ and $\{g_\mu\}$ be two homogenous bases of Λ. Then f and g are dual basis (i.e. $\langle f_\lambda, g_\mu \rangle = \delta_{\lambda\mu}$) if and only if $\sum_\lambda f_\lambda(x) g_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$.*

The homogeneity is just to make sure there are no issues about formal convergence of the sum; we could replace it with any condition that made the sum formally convergent.