LECTURE 3: COMPLETE HOMOGENOUS SYMMETRIC FUNCTIONS

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Define the **complete homogenous symmetric functions**

\[ h_k = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \]

**Example 1.**

\[ h_2 = \sum_{1 \leq i \leq j} x_i x_j = \sum_{i \geq 1} x_i^2 + \sum_{1 \leq i < j} x_i x_j = m_2 + m_{11} \]

For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \), define

\[ h_{\lambda} = h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_r} \]

**Example 2.**

\[ h_{21} = h_2 h_1 = (\sum_{i \geq 1} x_i^2 + \sum_{1 \leq i < j} x_i x_j)(\sum_{k \geq 1} x_k) \]

\[ = \sum_{i \geq 1} x_i^3 + 2 \sum_{i \neq j} x_i^2 x_j + 3 \sum_{1 \leq i < j < k} x_i x_j x_k \]

\[ = m_3 + 2m_{21} + 3m_{111} \]

For future reference,

\[ h_3 = m_3 + m_{21} + m_{111} \]
\[ h_{21} = m_3 + 2m_{21} + 3m_{111} \]
\[ h_{111} = m_3 + 3m_{21} + 6m_{111} \]

So we have a transformation matrix between \( h \)'s and \( m \)'s. For example, in degree 3, we just computed that the matrix is:

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6
\end{pmatrix}
\]

We have two main results today:

**Theorem 3.** The \( h \)'s form a basis of \( \Lambda \)

**Theorem 4.** The transformation matrix between \( h \)'s and \( m \)'s is symmetric.

1. **The \( h \)'s are a basis**

Since the \( h \) polynomials are the monomials in \( h_1, h_2, h_3, \ldots \), the claim is that \( \Lambda = \mathbb{Z}[h_1, h_2, h_3, \ldots] \).

We first show that the natural ring map \( \mathbb{Z}[h_1, h_2, \ldots] \to \Lambda \) is surjective. Indeed it is enough to show every \( e_k \) is a polynomial in \( h \)'s. Consider two identities

\[
\prod_{i \geq 1} (1 + x_i t) = \sum_{k \geq 0} e_k t^k
\]
\[
\prod_{i \geq 1} \frac{1}{1 - x_it} = \sum_{k \geq 0} h_k t^k
\]

Hence,
\[
1 = \prod_{i \geq 0} (1 - x_it) \prod_{i \geq 0} \frac{1}{1 - x_it} = (\sum_{k \geq 0} (-1)^k e_k t^k) (\sum_{k \geq 0} h_k t^k)
\]

Comparing the coefficients on both sides, one gets the relation:
\[
h_k - e_1 h_{k-1} + e_2 h_{k-2} - e_3 h_{k-3} + \ldots + (-1)^k e_k = 0 \quad (*)
\]

The dimension of degree \(d\) parts of \(\mathbb{Z}[h_1, h_2, \ldots]\) and \(\Lambda\) are equal, which proves the injectivity. \(\square\)

So \(\mathbb{Z}[h_1, h_2, \ldots] \cong \mathbb{Z}[e_1, e_2, \ldots] \cong \Lambda\).

1.1. What about \(\Lambda_n\)? We have \(\Lambda_n \cong \mathbb{Z}[e_1, e_2, \ldots]/I\) where \(I\) is the ideal generated by \(e_i = 0\) for \(i \geq n + 1\). If one thinks \(\Lambda \cong \mathbb{Z}[h_1, h_2, \ldots]\), then \(\Lambda_n \cong \mathbb{Z}[h_1, h_2, \ldots]/J\), where \(J\) is the coefficient of \(t^k\) in the power series expansion of \([\prod_{i \geq 0} (1 - x_it)]\) in \(t\) for \(k \geq n + 1\). The above proof can also show that \(\Lambda_n = \mathbb{Z}[h_1, h_2, \ldots, h_n]\) so \(\{h_\lambda : l(\lambda^T) \leq n\}\) is a basis for \(\Lambda_n\).
(In the lecture on Wednesday, we will show that \(\{h_\lambda : l(\lambda \leq n)\}\) is also a basis for \(\Lambda_n\)).

1.2. The map \(\omega\). (Section largely added by editor). Noticing the symmetric between \(e\) and \(h\), define a map \(\omega : \Lambda \to \Lambda\) by \(h_k \mapsto e_k\) and thus \(h_\lambda \mapsto e_\lambda\). By applying \(\omega\) to the equation \((*)\), one can get \(\omega(e_k) = h_k\), and hence \(\omega(e_\lambda) = h_\lambda\).

From our perspective, \(\omega\) is pretty mysterious. There are lots of applications of the ring of symmetric functions. In some of those other applications, \(\omega\) is more motivated. For example, if you use \(\Lambda\) to study the cohomology of the Grassmannian \(G(d, n)\), then \(\omega\) is the isomorphism \(G(d, n) \cong G(n-d, n)\) which sends a \(d\)-plane to its orthogonal complement. If you use \(\Lambda\) to study the representation theory of \(S_n\), then \(\omega\) tensors with the sign representation. There is not a similarly elegant answer for \(GL_n\) representation theory.

Let’s see what an answer would look like. First of all, what is the representation theory meaning of \(e_k\) and \(h_k\)? Remember that we go from a representation of \(GL_n\) to a symmetric polynomial by taking the trace of the action of a diagonal matrix. Let \(V\) be the standard \(n\)-dimensional representation of \(GL_n\). In \(V\), a diagonal matrix acts by itself, and thus has trace \(x_1 + x_2 + \cdots + x_n = e_1 = h_1\). Let’s look at \(\bigwedge^k V\). If \(V\) has basis \(e_1, e_2, \ldots, e_n\), then a basis for \(\bigwedge^k V\) is the \(\binom{n}{k}\) elements \(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\). A diagonal matrix of \(GL_n\) acts diagonally in this basis, with diagonal elements the products \(e_{i_1} e_{i_2} \cdots e_{i_k}\), and hence with trace \(e_k\). Similarly, \(\text{Sym}^k V\) will correspond to \(h_k\).

So we want an operation which switches \(\bigwedge^k V\) and \(\text{Sym}^k V\).

Such a thing occurs in physics and is called the “boson-fermion correspondence”; I don’t know much about it.

Such a thing also occurs in the theory of super-groups where, if \(V\) is the standard representation of \(GL_{-n}\), then \(\bigwedge^k V\) computed in the category of super-vector spaces is what we would normally call \(\text{Sym}^k V\). I also don’t know much about this, and we won’t talk about it.

In a month, we will talk about Schur-Weyl duality, which is about the relation between \(GL_n\) and \(S_n\) representation theory. Since \(\omega\) has a simple meaning on the symmetric group
side, we can try to use that to extract an interpretation for \( \omega \) on the \( GL \) side; let’s remember to think about that.

**Question from the floor:** Are you saying there is no functor \( GL - \text{rep} \to GL - \text{rep} \) which realizes \( \omega \)? **Answer:** No, I am not willing to make such a specific claim. I am saying it is not any familiar or elegant operation.

2. **The matrix is symmetric**

Define \( A_{\lambda \mu} = \text{coefficient of } m_\mu \text{ in } h_\lambda \).

**Theorem 5.** \( A_{\lambda \mu} = A_{\mu \lambda} \).

**Proof.** To get output \( x_1^{\mu_1}x_2^{\mu_2} \ldots x_r^{\mu_r} \) the \( h_\lambda \) term must contribute \( x^{\alpha_j} = \prod_k x_k^{\alpha_j(k)} \), where \( \alpha_j \)'s are vectors in \( \mathbb{Z}_{\geq 0} \) with \( \sum_j \alpha_j = \mu \) and \( \sum_k \alpha_j(k) = \lambda_j \) for all \( j \). Thus,

\[
A_{\lambda \mu} = \# \left\{ (\alpha_1, \ldots, \alpha_r) \mid \sum_k \alpha_j(k) = \lambda_j, \sum_j \alpha_j = \mu \right\}
\]

Which is equivalent to say

\[
A_{\lambda \mu} = \text{number of nonnegative integer matrices with row sum } \mu \text{ and column sum } \lambda
\]

Clearly, nonnegative integer matrices with row sum \( \mu \) and column sum \( \lambda \) is in bijection with those with row sum \( \lambda \) and column sum \( \mu \). Therefore, \( A_{\lambda \mu} = A_{\mu \lambda} \). \( \square \)

**Example 6.** \( \Lambda = (2, 1), \mu = (1, 1, 1) \), all possible such matrices are

\[
\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

This shows that the coefficient of \( m_{111} \) in \( h_{21} \) should be 3.

We can write this proof using generating function identities. By definition, we have

\[
\sum_{\lambda, \mu} A_{\lambda \mu} m_\lambda(x)m_\mu(y) = \sum_\lambda m_\lambda(x)h_\lambda(y)
\]

and the above proof showed that

\[
\sum_{\lambda, \mu} A_{\lambda \mu} m_\lambda(x)m_\mu(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}
\]

The right hand side is sometimes known as the **Cauchy product**. Similarly,

\[
\sum_{\mu, \lambda} A_{\mu \lambda} m_\lambda(x)m_\mu(y) = \sum_\mu h_\mu(x)m_\mu(y) = \prod_{i,j \geq 1} \frac{1}{1 - y_i x_j}
\]

The right hand sides are equal, so so are the left hand sides, showing \( A_{\lambda \mu} = A_{\mu \lambda} \).

Let’s explain the identity \( \sum_{\lambda, \mu} A_{\lambda \mu} m_\lambda(x)m_\mu(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} \) more slowly. Expand the geometric series in the Cauchy product

\[
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \left(1 + x_1 y_1 + x_1^2 y_1^2 + \ldots \right) \left(1 + x_2 y_2 + x_2^2 y_2^2 + \ldots \right) \ldots
\]

\[
\left(1 + x_2 y_1 + x_2^2 y_1^2 + \ldots \right) \left(1 + x_2 y_2 + x_2^2 y_2^2 + \ldots \right) \ldots
\]

\[
\ldots
\]
The coefficient of $x_1^7x_2^3y_1^5y_2^2$ is the number of ways in which we pick $(x_i, y_j)^k$ such that the powers of $x_i$’s is $(7, 3)$, and powers of $y_j$’s is $(5, 5)$. This is in bijection with the integer matrices with nonnegative entries with row sum $(7, 3)$ and column sum $(5, 5)$. For example $(x_1y_1)^5(x_1y_2)^3(x_2y_2)^3$ corresponds to the matrix $(\frac{5}{3}, \frac{2}{3})$.

2.1. **The Hall inner product.** The symmetry of the matrix $A_{\lambda\mu}$ can be used in the following way: Define a bilinear product $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, where $\delta_{\lambda\mu}$ is the Kronecker delta, and then $\langle h_\lambda, h_\mu \rangle = \langle h_\lambda, \sum_\nu A_{\lambda\nu} m_\nu \rangle = A_{\lambda\mu}$. Since $A_{\lambda\mu} = A_{\mu\lambda}$, this bilinear form is symmetric.

Here is a useful fact about this bilinear form, which we’ll prove next time.

**Proposition 7.** Let $\{f_\lambda\}$ and $\{g_\mu\}$ be two homogenous bases of $\Lambda$. Then $f$ and $g$ are dual basis (i.e. $\langle f_\lambda, g_\mu \rangle = \delta_{\lambda\mu}$) if and only if

$$\sum_\lambda f_\lambda(x)g_\lambda(y) = \prod_{i,j} \frac{1}{1-x_iy_j}.$$  

The homogeneity is just to make sure there are no issues about formal convergence of the sum; we could replace it with any condition that made the sum formally convergent.