NOTES FOR SEPTEMBER 19, 2012: SEMI-STANDARD YOUNG TABLEAUX

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(Monday is Rosh Hoshanah. Jonah Blasiak will substitute on Monday. You can email your homework solutions or put homework solutions in David Speyer's mailbox.)

Today we will introduce Schur polynomials. They are the most important basis for symmetric polynomials and they are the characters of $\mathrm{GL}_n(\mathbb{C})$, but they are difficult to define. We will start by describing them in a combinatorial way, via semi-standard young tableaux.

1. Definition and First Examples

Definition 1.1. For a partition λ, a *semi-standard young tableaux (SSYT)* of shape λ is a filling of the basis of λ with positive integers so that the rows weakly increase and the columns strictly increase.

Remark. *Tableaux* is plural, *tableau* is singular. If you know this, you may avoid annoying a Francophone referee. Or you can just always write SSYT.

Example 1.1. For $\lambda = (4, 2, 1)$, a SSYT of shape λ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

Definition 1.2. If T is a SSYT, then we write

$$x^T = x_1^{\text{number of 1's}} x_2^{\text{number of 2's}} \cdots .$$

So to each SSYT, we may assign a monomial in x_i's.

Example 1.2. We have

$$x^{\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}} = x_1^2 x_2^2 x_3 x_5 x_6.$$

Definition 1.3. We define the *Schur polynomial of shape* λ to be

$$s_\lambda(x_1, x_2, \ldots) = \sum_{T \text{ of shape } \lambda} x^T.$$

Remark. It is not clear that s_λ is a symmetric polynomial.

Example 1.3. Consider s_{21}. We could have

$$\begin{array}{|c|c|}\hline i & i \\ \hline j \\ \cline{1-1}\end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline i & j \\ \hline j \\ \cline{1-1}\end{array} \qquad \text{for } i < j,$$

or

$$\begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array} \quad \text{or} \quad \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array} \qquad \text{for } i < j < k.$$

Therefore, we have

$$s_{21} = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 + 2 \sum_{i<j<k} x_i x_j x_k = m_{21} + 2m_{111}.$$

Example 1.4. We have $s_k = h_k$ because $i_1 \le \cdots \le i_k$ is the only SSYT of shape k. So

$$s_k = \sum x_{i_1} \cdots x_{i_k} = h_k.$$

Example 1.5. We have $s_{\underbrace{11\cdots1}_{k}} = e_k$ since $i_1 < \cdots < i_k$ is the only tableau of shape 1^k. Hence

$$s_{1^k} = \sum x_{i_1} \cdots x_{i_k} = e_k.$$

Example 1.6. In Λ_n, we have

$$s_{(\lambda_1+1)(\lambda_2+1)\cdots(\lambda_n+1)} = x_1 \cdots x_n s_{\lambda_1 \cdots \lambda_n}.$$

This is because the left hand column of a tableau of shape $(\lambda_1+1)(\lambda_2+1)\cdots(\lambda_n+1)$ must have first column $123\cdots n$. Note that if $\ell(\lambda) > n$, then $s_\lambda = 0$ in Λ_n.

2. Why is S_λ symmetric?

Because the symmetric group is generated by transpositions, it is enough to show that the coefficients of $x_1^{a_1} \cdots x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}$ and $x_1^{a_1} \cdots x_i^{a_{i+1}} x_{i+1}^{a_i} \cdots x_n^{a_n}$ are equal.

Consider the tableau T and look only at the positions of i and $i+1$. Set $j = i+1$. Now consider a portion of a tableau:

$$\begin{ytableau} * & * & * & * & * & * & i & i & i & i & j & j & j & j \\ * & * & i & i & i & j & j & j \\ i & i & j \end{ytableau}$$

If an i and a j are in the same column, pair them off. The remaining i's and j's lie in horizontal strips, where each strip looks like

$$\begin{ytableau} i & i & i & j & j \end{ytableau}$$

Within each row, interchange the number of i's and j's. Then we have

$$\begin{ytableau} i & i & j & j & j \end{ytableau}$$

The total effect is that we will have switched the number of i's and j's. This gives an involution

$$S_i \colon \mathrm{SSYT}(\lambda) \to \mathrm{SSYT}(\lambda),$$

switching the number of i's and the number of $j = (i+1)$'s. This involution is called the *Bender-Knuth involution.*

In particular, this takes a tableau with corresponding monomial $x_1^{a_1} \cdots x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n}$ to a tableau with monomial $x_1^{a_1} \cdots x_i^{a_{i+1}} x_{i+1}^{x_i} \cdots x_n^{a_n}$. This therefore show that s_λ is symmetric. $\square$

3. Why are the S_λ a basis?

We first introduce some notation.

Definition 3.1. Write $s_\lambda = \sum K_{\lambda\mu} m_\mu$. So $K_{\lambda\mu}$ is the number of SSYT of shape λ with content μ. The $K_{\lambda\mu}$ are called *Kostka numbers.*

Example 3.1. The content of

$$\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}$$

is $(3,1)$ since

$$x^{\begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau}} = x_1^3 x_2^1.$$

On Problem Set 2, Problem 3, we will show that it $K_{\lambda\mu}$ is nonzero, then $\lambda \succeq \mu$. That is, if $\lambda \not\succeq \mu$, then $K_{\lambda\mu}$ and if $\lambda = \mu$ then $K_{\lambda,\mu}$. Using this, we have that the s_λ's are upper triangular in the m_μ's with 1's along the diagonal. Thus s_λ is a basis for Λ.

In Λ_n, the s_λ with $\ell(\lambda) \le n$ are a basis and $s_\lambda = 0$ if $\ell(\lambda) > n$.

4. The Orthonormality of s_λ

We mention two ways of doing this. Recall that we defined $\langle\ ,\ \rangle$ so that $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$. In matrices, we can check one of two things:

(1) We can check that the matrices $m \to s$ and $s \to h$ are transposes of each other. That is, we can check $\sum K_{\mu\lambda} s_\lambda = h_\mu$. (This is the route that Stanley's book uses.) In the end, this comes down to wanting to show that there is a bijection between rectangular arrays of integers ($m \to h$ matrix) to paris of SSYT. There is such a bijection and this bijection is called the *Robinson-Schensted-Knuth algorithm (RSK)*.

More precisely, given λ, μ with $|\lambda| = |\mu|$, the number of rectangular arrays with row sum λ and column sum μ is equal to the number of ordered pairs (T, U) where T and U are SSYT of the same shape with content$(T) = \lambda$ and content$(U) = \mu$.

(2) The other strategy is to show that $h \to s$ and $s \to m$ are transposes of each other. We will do this next week. We need formulas for writing these bases in terms of each other. We defer the proofs to next week but state the formulas now.

The formula $h \to s$ is called *Jacobi-Trudi* and says

$$s_{\lambda_1\cdots\lambda_n} = \det\begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+n-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_n-n-1} & h_{\lambda_n-(n-2)} & \cdots & h_{\lambda_n} \end{pmatrix}$$

So for example,

$$s_{31} = \begin{vmatrix} h_3 & h_4 \\ h_0 & h_1 \end{vmatrix}.$$

Here $h_0 = 1$ and $h_{-k} = 0$ for $k > 0$.

The formula $s \to m$ is a ratio of alternants

$$s_{\lambda_1\cdots\lambda_n}(x_1, \dots, x_n) = \frac{\det\begin{pmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \cdots & x_n^{\lambda_1+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_{n-1}+1} & x_2^{\lambda_{n-1}+1} & \cdots & x_n^{\lambda+n-1+1} \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{pmatrix}}{\det\begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_n \\ 1 & 1 & \cdots & 1 \end{pmatrix}}$$

Alternatively, the denominator is $\prod_{i<j}(x_i - x_j)$.