NOTES FOR SEPTEMBER 17: JACOBI-TRUDI IDENTITY

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Our goal today is to express s_λ in terms of $\{h_\mu : \mu \vdash n\}$. We do so via the following:

Theorem 1. *(Jacobi-Trudi Identity): For λ satisfying $\ell(\lambda) \leq n$, we have*

$$s_\lambda = \det((h_{\lambda_i - i + j})_{i,j \in [n]}).$$

Here, $h_0 = 1, h_k = 0$ for $k < 0$.

We will refer to this henceforth as JT.

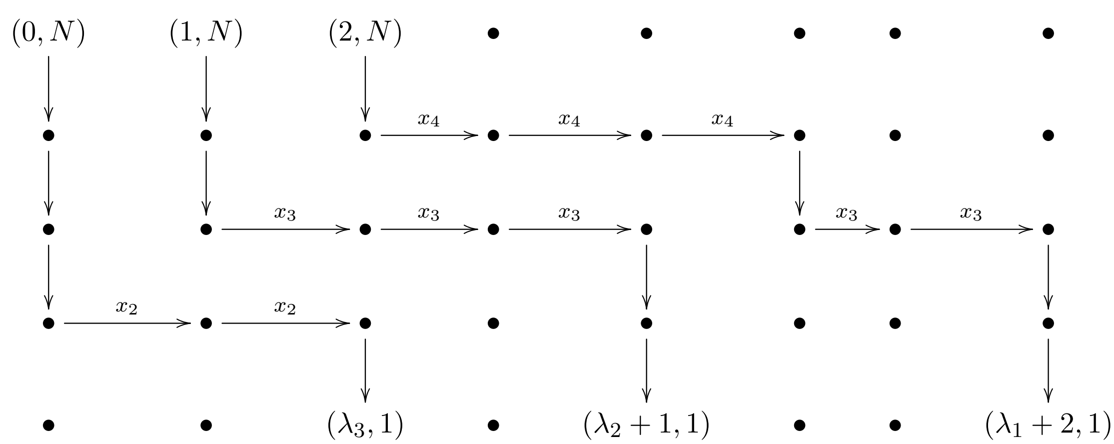
Example 1. JT says that

$$\begin{aligned} s_{321} &= \det \begin{pmatrix} h_3 & h_4 & h_5 \\ h_1 & h_2 & h_3 \\ 0 & h_0 & h_1 \end{pmatrix} \\ &= h_{321} - h_{411} - h_{330} + h_{510}. \end{aligned}$$

The way to "remember" the statement of the theorem is to note that the indices of the the h_i's along the diagonal are the parts of λ, and that the indices increase by 1 as they move right. For example: we have h_3, h_2, h_1 along the diagonal in the preceding example.

Proof. Fix some $N >> 0$. Let the "top vertices" denote the n integer lattice points with coordinates $(0, N), (1, N), \ldots, (n-1, N)$. Let the "bottom vertices"' denote the n points with coordinates $(\lambda_n, 1), (\lambda_{n-1}+1, 1), (\lambda_{n-2}+2, 1), \ldots, (\lambda_1 + (n-1), 1)$. Read these bottom points from left to right, so that $(\lambda_n, 1)$ is the first point, and $(\lambda_1 + n - 1, 1)$ is the last point. In the figure below, we have taken $n = 3$ and $N = 5$. The top vertices and bottom vertices are the labeled lattice points. λ is the partition $\lambda = (532)$.

We want to interpret both the LHS and RHS of JT as weighted sums of collections of n lattice paths joining the "top vertices" to the "bottom vertices". Namely, we are considering ***lattice n-paths***, $L = (L_1, \ldots, L_n)$, joining the top points to the bottom points. Each L_i is a lattice path joining $(i-1, N)$ to the $\sigma(i)^{th}$ special point on the bottom, for some $\sigma \in \mathfrak{S}_n$. Each L_i is only allowed to move down or right. The figure below shows a lattice 3-path with $\sigma = \text{id}$.



We weight horizontal edges $(k, j) \to (k+1, j)$ in the integer lattice with the monomial x_j, and give all vertical edges have weight 1. Then the ***weight*** of a given path L_j is the product of its weights, which is

$$w(L_j) := \prod_{\text{horizontal steps of } L_j \text{ at height } i} x_i.$$

The ***weight*** of a lattice n-path, $L = (L_1, \ldots, L_n)$ is just the product of the weights of its paths:

$$w(L) := \prod_i w(L_i).$$

The weights of $L(1), L(2), L(3)$ in the diagram above are x_2^2, x_3^3 and $x_4^3 x_3^2$ respectively. $w(L)$ is consequently $x_2^2 x_3^5 x_4^3$.

Now let's return to the proof.

Claim 1. For a fixed $\sigma \in \mathfrak{S}_n$:
a)

$$\sum_{L \text{ joins top to bottom according to } \sigma} w(L) = \prod_{i=1}^{n} h_{\lambda_{\sigma(i)} - \sigma(i) + i}.$$

b) The RHS of JT is

$$\sum_{L=(L_1,\ldots,L_n)} \epsilon_{\sigma(L)} w(L).$$

To see a): First, we note that

$$\sum_{L_j \text{ has horizontal distance } k} w(L_j) = h_k.$$

Indeed, by definition,

$$h_k(x_1, \ldots, x_n) = \sum_{i_1 \leq i_2 \leq i_n} x_1^{i_1} \cdots x_n^{i_n}$$

and we associate to the sequence $(i_1 \leq i_2 \leq i_n)$ the path that takes i_n steps right, then moves down 1, then takes i_{n-1} steps right, then moves down 1, etc. In the end, we move k steps right. The resulting path has weight $x_1^{i_1} \cdots x_n^{i_n}$.

Once you know this, observe that paths of type σ have L_i moving $\lambda_{\sigma(i)} - \sigma(i) + i$ steps right, from which we obtain a), since the weight of the collection $(L_1, \ldots, L_n)$ is just the product of their individual weights.

To see b): Follows from the definition of the determinant as a sum over permutations, using part a).

Claim 2. $s_\lambda = \sum_L \epsilon_{\sigma(L)} w(L) = \text{RHS of JT}$, by Claim 1.

To do this, we note that n-lattice paths in which some L_i intersects another L_j cancel out in pairs. Let's do this carefully: pick the topmost and leftmost point of intersection of L, and call the two paths that are crossing L_i and L_j. We define a new lattice n-path $\tilde{L}$ as follows. Every path in L besides paths L_i and L_j becomes a path in $\tilde{L}$. We must specify two more paths to complete our description of $\tilde{L}$. In the first path, we follow L_i until the point of intersection, after which we follow L_j. In the second path, we follow L_j until the intersection, after which we follow L_i. Then $w(L) = w(\tilde{L})$ since all of the same edges are taken. However the the permutations associated to the two paths differs by the transposition (ij), so that their signs are opposite, and the contributions of L and $\tilde{L}$ to the above weighted sum cancel out.

The remaining paths are the *nonintersecting* lattice n-paths. Note that any such lattice n-path corresponds to the identity permutation ("topologically" obvious: any nontrivial permutation would have an inversion, and the two paths corresponding to this inversion would have to cross). The drawing of a 3-lattice path we gave above is an example of such a nonintersecting path.

We claim nonintersecting lattice n-paths of shape λ are in weight-preserving bijection with ***reverse SSYT*** of shape λ. A ***reverse SSYT*** is a filling of λ that is weakly decreasing in rows and strictly decreasing in columns. We fill the first row of λ by reading off the horizontal edges of L_n in decreasing order. The second row of λ is filled with the horizontal edges of L_{n-1}, and so on. For example, the 3-lattice path we drew above corresponds to the Reverse SSYT

$$\begin{array}{|c|c|c|c|c|}\hline 4 & 4 & 4 & 3 & 3 \\ \hline 3 & 3 & 3 \\ \cline{1-3} 2 & 2 \\ \cline{1-2} \end{array} .$$

Clearly, the filling we have prescribed is decreasing along rows. That it is stricly decreasing in columns amounts to the assertion that the paths do not self-intersect. This process is reversible, so we have the claimed bijection.

By the preceding, we will have proven claim 2, and consequently the JT theorem, after the following exercise:

Exercise: $s_\lambda = \sum_{T \in \text{ReverseSSYT}(\lambda)} x^T$. (Hint, fill each box in a reverse SSYT by n minus what is currently in the box).

□

Remark: Jacobi-Trudi generalizes to skew shapes:

$$s_{\lambda/\mu} = \det((h_{\lambda_i - \mu_i - i + j})_{i,j \in [n]}).$$

Here, the definition of $s_{\lambda/\mu}$ is the same as the combinatorial definition of the s_λ: it is the content-generating function for SSYT of shape λ/μ. The proof of this identity is the same as above, where one uses the parts of μ to define the values of the "top points", just as we used the value of λ to define the "bottom points" in our proof above.

Remark: I believe we have not yet discussed that the involution ω sends $s_\lambda \mapsto s_{\lambda'}$. Once we see this, we will obtain as a corollary

$$s_\lambda = \det((e_{\lambda_i^T} - i + j)_{i,j \in [n]}),$$

which follows by applying ω to both sides of JT.

David adds: Actually, my plan was to have Jonah also mention the dual identity stated here, point out that it can be roved the same way, and deduce from that that ω sends s_λ to s_{λ^T}.