NOTES FOR SEPTEMBER 12

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Last time:

- We defined an inner product $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$ and showed that this puts a symmetric bilinear form on $\Lambda \times \Lambda \to \mathbb{Z}$. On the homework, you will check that it is positive definite.
- We claimed that the h_λ with $\ell(\lambda) \leq n$ are a basis for Λ_n.

Today: we'll check this claim, and also show that if (f_λ), (g_λ) are families of symmetric polynomials such that $\sum_\lambda f_\lambda(x)g_\lambda(y) = \prod_{i,j} \frac{1}{1-x_iy_j}$, then $\{f_\lambda\}$ and $\{g_\lambda\}$ are dual bases with respect to the inner product $\langle \cdot, \cdot \rangle$, that is, $\langle f_\lambda, g_\mu\rangle = \delta_{\lambda\mu}$.

Claim. The h_λ with $\ell(\lambda) \leq n$ are a basis for Λ_n.

Proof. Given our inner product on Λ and a surjective map $\pi : \Lambda \to \Lambda_n$, we can build a backwards map $\Lambda_n \to \Lambda$, since Λ decomposes as $\Lambda = \ker\pi \oplus (\ker\pi)^\perp$, and the orthogonal complement is then isomorphic to Λ_n. Note that we have used the positive-definiteness of our inner product to get the direct sum decomposition: in particular, positive definiteness implies that $(\ker\pi) \cap (\ker\pi)^\perp = 0$, so that $\ker\pi + (\ker\pi)^\perp = \ker\pi \oplus (\ker\pi)^\perp = \Lambda$.

Our map $\pi : \Lambda \to \Lambda_n$ is defined by sending

$$x_i \mapsto \begin{cases} x_i & i \leq n \\ 0 & i > n. \end{cases}$$

In our setting, $\ker\pi = \mathrm{Span}(m_\lambda \;:\; \ell(\lambda) > n)$, so (using the dual basis) the orthogonal complement is $(\ker\pi)^\perp = \mathrm{Span}(h_\lambda \;:\; \ell(\lambda) \leq n)$. In particular, this shows that the (span of the) h_λ maps isomorphically to Λ_n. □

Note: Our preferred lifting $\Lambda_n \to \Lambda$ will be $h_\lambda \mapsto h_\lambda$ for $\ell(\lambda) \leq n$. We will use this to define an inner product on Λ_n.

Theorem. Let f_λ, g_λ be sets of homogeneous symmetric polynomials such that $\deg f_\lambda = \deg g_\lambda = |\lambda|$. If $\displaystyle\sum_\lambda f_\lambda(x)g_\lambda(y) = \prod_{i,j}\frac{1}{1-x_iy_j}$, then f and g are dual bases (with respect to our inner product $\langle \cdot, \cdot \rangle$.

Proof. . First we show the f_λ span $\mathbb{Q} \otimes \Lambda$. If not, there exists $h \neq 0$ in Λ such that $\langle f_\lambda, h\rangle = 0$ for all λ. We show that $h = 0$.

Write h in the homogeneous basis, $h(x) = \sum c_\lambda h_\lambda(x)$. We have

$$\begin{aligned}\langle h(x), \textstyle\prod \frac{1}{1-x_iy_j}\rangle &= \langle h(x), \textstyle\sum h_\lambda(y)m_\lambda(x)\rangle \\ &= \textstyle\sum c_\lambda h_\lambda(y)\end{aligned}$$

(The inner product is in the x variables.) On the other hand, $\langle h(x), \sum f_\lambda(x)g_\lambda(y)\rangle = 0$. So, we can conclude that $\sum c_\lambda h_\lambda(y) = 0$, so by linear independence $c_\lambda = 0$. Thus h was zero after all.

Thus the f_λ span $\mathbb{Q} \otimes \Lambda$ in every degree, and by dimension counting (since we assumed $\deg f_\lambda = |\lambda|$, the f_λ's are indexed by partitions, just like the known e, h, m bases) they are linearly independent. Since this holds in each degree, we see that the f_λ collectively give a basis for $\mathbb{Q} \otimes \Lambda$. So there exists *some* dual basis $f_\lambda^\vee$. We will show that $f_\lambda^\vee = g_\lambda$.

First consider the inner product

$$\langle f_\lambda^\vee(x), \prod_{i,j}\frac{1}{1-x_iy_j}\rangle = \langle f_\lambda^\vee(x), \sum f_\lambda(x)g_\lambda(y)\rangle = g_\lambda(y).$$

On the other hand,

$$\langle m_\lambda(x), \prod_{i,j} \frac{1}{1-x_iy_j}\rangle = \langle m_\lambda(x), \sum h_\lambda(x)m_\lambda(y)\rangle = m_\lambda(y).$$

So, by linearity, for any $f \in \Lambda$,

$$\langle f(x), \prod_{i,j} \frac{1}{1-x_iy_j}\rangle = f(y).$$

Thus, our previous equation now reads

$$\langle f_\lambda^\vee(x), \prod_{i,j} \frac{1}{1-x_iy_j}\rangle = g_\lambda(y) = f_\lambda^\vee(y). \qquad \square$$

Note: Our assumptions of homogeneity (and symmetry) were important in this theorem. Here's an example of why: in the case $n = 1$, we have

$$\frac{1}{1-xy} = \sum h_k(x)m_k(y) = \sum x^ky^k.$$

Now we'll make some 'bad choices'. In $\Lambda \otimes \mathbb{Q}$, we could write

$$\frac{1}{1-xy} = \sum h_k(x)m_k(y) = 1 + \underbrace{(\tfrac{1}{2}x)(\tfrac{1}{2}y) + \cdots + (\tfrac{1}{2}x)(\tfrac{1}{2}y)}_{\text{4 times}} + \underbrace{(\tfrac{1}{2}x^2)(\tfrac{1}{2}y^2) + \cdots + (\tfrac{1}{2}x^2)(\tfrac{1}{2}y^2)}_{\text{4 times}} + \cdots$$

So now we have

$$\begin{array}{llll} f_0 & = 1 & g_0 & = 1 \\ f_1 & = f_2 = f_3 = f_4 = \frac{1}{2}x & g_1 & = g_2 = g_3 = g_4 = \frac{1}{2}y \\ f_5 & = f_6 = f_7 = f_8 = \frac{1}{2}x^2 & g_5 & = g_6 = g_7 = g_8 = \frac{1}{2}y^2 \\ f_9 & = \cdots = f_{12} = \frac{1}{2}x^3 & g_9 & = \cdots = g_{12} = \frac{1}{2}y^3 \end{array}$$

So the f's are too numerous to be a basis for $\mathbb{Q} \otimes \Lambda$. (We could make similar counterexamples without working over $\mathbb{Q}$, just by choosing 'too many' polynomials in each degree.)

Analogous statement for finite-dimensional vector spaces. Say V is a finite dimensional vector space with basis $v_1, \ldots, v_n$ and $V^\vee$ its dual, with basis $w_1, \ldots, w_n$. Then (v_i) and (w_i) are dual bases if and only if

$$\sum_{i=1}^n v_i \otimes w_i = \text{Id} \text{ in } V \otimes V^\vee = \mathcal{H}om(V, V).$$

In our setting, we have the identity $k[x, y] = k[x] \otimes_k k[y]$ (as k-algebras), and we've been using this to dodge writing $\otimes$, and using the inner product $\langle \cdot, \cdot \rangle$ to dodge distinguishing V and $V^\vee$.

Next lecture (Friday): Schur polynomials s_λ, our last basis for Λ. These appear as characters of irreducible representations of GL_n and are ***self-dual***: $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$. Bear in mind that it isn't (a priori) obvious that a self-dual basis exists! (Note that one must exist in $\Lambda \otimes \mathbb{R}$, since there's only one positive definite bilinear form on $\mathbb{R}$.) On the other hand, since it exists, it is (essentially) unique:

Observation: Let $L \cong \mathbb{Z}^n$ be a free abelian group of rank n with a symmetric bilinear form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$. Suppose there exists a basis e_i for L such that $\langle e_i, e_j \rangle = \delta_{ij}$. Then any other such basis must be of the form $\{\pm e_i\}$.

Proof. Let f_i be another orthonormal basis, and write $f_i = \sum c_i e_i$, with $c_i \in \mathbb{Z}$. Then

$$\langle f_i, f_i \rangle = \sum c_i^2 = 1,$$

so one $c_i = \pm 1$ and all the others are 0. Thus $f_i = \pm e_j$ for some j. Since $\langle f_i, f_j \rangle = 0$ for $i \neq j$, the f_i are a permutation of the $(\pm)e_i$. $\square$

Monday and Wednesday: Fill in the details and prove two big identities needed to prove the self-duality of the Schur polynomials s_λ.