NOTES FOR SEPTEMBER 19, 2012: THE RATIO OF ALTERNANTS FORMULA

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Today we will we begin the proof of the ratio of alternants formula for schur polynomials.

1. Beginning Examples and Discussion

A Schur polynomial:

$$\begin{array}{cccccc}
 & & xy^2+ & & x^2y & \\
s_{21}(x,y,z) = & y^2z+ & & 2xyz+ & & x^2z \\
 & & yz^2+ & & xz^2 &
\end{array}$$

The coefficients of $s_{72}(x,y)$:

$$\begin{array}{ccccccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 2 & & 2 & & 2 & & 2 & & 2 & & 1 & \\
1 & & 2 & & 3 & & 3 & & 3 & & 3 & & 2 & & 1 \\
 & 1 & & 2 & & 3 & & 3 & & 3 & & 2 & & 1 & \\
 & & 1 & & 2 & & 3 & & 3 & & 2 & & 1 & & \\
 & & & 1 & & 2 & & 3 & & 2 & & 1 & & & \\
 & & & & 1 & & 2 & & 2 & & 1 & & & & \\
 & & & & & 1 & & 1 & & 1 & & & & &
\end{array}$$

In both of the diagrams, the page is considered to be the plane $a+b+c = |\lambda|$ in $\mathbb{Z}^3$. The monomial $x^a y^b z^c$ in the schur polynomial is associated with the point (a, b, c). In the second diagram, only the coefficients of the monomials are given.

Question from the floor: Why do we get nice shapes, like hexagons?

Answer You know that, if the monomial m_μ appears in s_λ, the $\mu \preceq \lambda$. So what we are seeing is the set of μ such that $\mu \preceq \lambda$, for a fixed λ. The elegant geometry of these pictures can be explained by:

Proposition Let λ and μ be partitions with n parts. Then $\mu \preceq \lambda$ if and only if the vector μ in $\mathbb{Z}^n$ is in the convex hull of $S_n \cdot \lambda$.

We will prove one direction of this. Assume $\mu \in \mathrm{Hull}(S_n \cdots \lambda)$. We want to show $\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$. A linear function on a convex bounded polytope is always maximized at a vertex. $\max\limits_{x \in \mathrm{Hull}(S_n \cdot \lambda)} (x_1 + \cdots + x_k) = \max\limits_{x \in S_n \cdot \lambda} (x_1 + \cdots + x_k) = \max\limits_{i_1, \cdots, i_k \text{distinct}} (\lambda_{i_1} + \cdots + \lambda_{i_k}) \leq \lambda_1 + \cdots + \lambda_k$. So in particular $\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$.

You might also notice that the coefficients in these pictures are varying in a piecewise linear way. See Problem Set 2, Problem 2.

2. The Ratio of Alternants Formula – start of proof

Notation

- For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, let $\Delta(\lambda) = \det(x_i^{\lambda_j}) = \det \begin{pmatrix} x_1^{\lambda_1} & x_2^{\lambda_1} & \cdots & x_n^{\lambda_1} \\ x_1^{\lambda_2} & x_2^{\lambda_2} & \cdots & x_n^{\lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{pmatrix}$.
- Let $\rho = (n-1, n-2, \cdots, 1, 0)$.

Theorem(Ratio of alternants formula): $s_\lambda(x_1, x_2, \cdots, x_n) = \frac{\Delta(\lambda+\rho)}{\Delta(\rho)}$.

$\Delta(\lambda)$ is antisymmetric in the x_i, so $x_i - x_j$ divides $\Delta(\lambda)$ for all $i < j$. This means that $\prod_{i<j}(x_i - x_j)$ divides $\Delta(\lambda)$ for all λ. Set $F_\lambda(x_1, \cdots, x_n) = \frac{\Delta(\lambda)}{\prod_{i<j}(x_i - x_j)}$. The numerator and denominator are both antisymmetric in the x_i, so F_λ is symmetric in the x_i. We also have that $\deg(F_\lambda) = |\lambda| - \binom{n}{2}$.

In particular, F_ρ has degree 0, so it is constant; in fact it is 1. Let's state this explicitly:

$$\det \begin{pmatrix} x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \det(x_i^{n-j}) = \Delta(\rho) = \prod_{i<j}(x_i - x_j).$$

This is called ***Vandermonde's Determinant***.

To prove the ratio of alternants formula, we need to show that $s_\lambda \cdot \Delta(\rho) = \Delta(\lambda + \rho)$. So we want to prove

$$\left(\sum_{\mu \in \mathbb{Z}^n_{\geq 0}} K_{\lambda\mu} x^\mu\right)\left(\sum_{\omega \in S_n} (-1)^\omega x^{\omega(p)}\right) = \sum_{\omega \in S_n} (-1)^\omega x^{\omega(\lambda+\rho)} \tag{1}$$

Notation

- S_n is the group of permutations of n letters.
- For $\omega \in S_n$, $(-1)^\omega$ is the sign of the permutation.
- S_n acts on $\mathbb{Z}^n$ by permuting the entries.
- For a partition λ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $K_{\lambda\alpha}$ is the number of SSYT of shape λ and content α.
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots, \alpha_n)$

We want to show that the coefficient of x^β is the same on both sides of (1). Both sides of (1) are antisymmetric, so if we know that the coefficients are equal when $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ then we can use the antisymmetry to get that the coefficients are equal for all β. We also have, by the antisymmetry, that if $\beta_i = \beta_{i+1}$ then both coefficients are 0, so we may assume that $\beta_1 > \beta_2 > \cdots > \beta_n$.

Define α by $\beta = \alpha + \rho$. The entries of β are strictly decreasing, so α is a partition. Every monomial appearing in the expansion on both sides of (1) has degree $|\lambda| + |\rho|$. This means we are interested in when $|\beta| = |\lambda| + |\rho|$, so $|\alpha| = |\lambda|$. Matching the coefficients of $x^{\alpha+\rho}$ on both sides of (1), we get that we need

$$\sum_{w \in S_n} (-1)^\omega K_{\lambda, \alpha+\rho-w(\rho)} = \begin{cases} 1 & \alpha = \lambda \\ 0 & \text{otherwise} \end{cases}$$

To achieve this end we will use a trick introduced by Eğecioğlu and Remmell[1]. (At least, David thinks they were the first to use this trick.)

Because $K_{\lambda\gamma}$ is a symmetric function of γ, we can change the left hand side of this equation to

$$\sum_{w \in S_n} (-1)^w K_{\lambda, w^{-1}(\alpha)+w^{-1}(\rho)-\rho} = \sum_{v \in S_n} (-1)^v K_{\lambda, v(\alpha)+v(\rho)-\rho} = \sum_{v \in S_n} (-1)^v K_{\lambda, v^*(\alpha)}.$$

Here we define

$$v^*(\alpha) := v(\alpha) + v(\rho) - \rho.$$

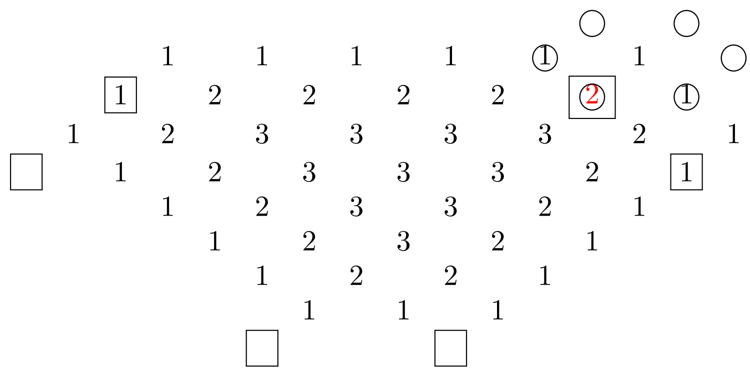
[1]Eğecioğlu and Remmel, *A combinatorial interpretation of the inverse Kostka matrix*, Linear and Multilinear Algebra **26** (1990), no. 1-2, 59–84.

Our goal is now to prove that

$$\sum_{v \in S_n} (-1)^v K_{\lambda, v^*(\alpha)} = \begin{cases} 1 & \alpha = \lambda \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The advantage of this change of variables is that $\alpha \mapsto v^*(\alpha)$ is an action of S_n. Specifically, it is the standard S_n action translated to fix $-\rho$ instead of 0.

Example We show the effect of this trick for $\lambda = (7,3,0)$, $\alpha = (6,3,1)$. The coefficients of s_{730} are displayed as in the beginning of the notes. The coefficient of x^α is marked in red. The terms of the form $\alpha + \rho - w(\rho)$ are circled; those of the form $v(\alpha) + v(\rho) - \rho$ are surrounded by squares. In both cases, we are trying to prove that an alternating sum of the marked numbers is zero, namely, that $2 - 1 - 1 = 0$. Notice that the circled numbers form a small hexagon near α, because they are all displacements of α by vectors $\rho - w(\rho)$. On the other hand, notice that the squared numbers form a hexagon with center slightly offset from the center of the figure; this is because the starred action is a translate of the standard action. Finally, notice that each square corresponds to a circle which contains the same number, in a position related to it by the non-starred S_3 action.



3. Preview of next class

We will prove (2) next class by using a sign canceling involution. We will always be canceling tableaux with content γ against tableaux of content $s_i^*\gamma$ where $s_i = s_{i,i+1} \in S_n$ is the permutation switching i and $i+1$. Since the starred maps give an S_n action, $s_i^* v^* \alpha = (s_i v)^* \alpha$, so we are canceling terms of the correct forms.

So, let's understand the action of s_i^*. We have

$$s_i^*(\gamma_1, \cdots, \gamma_2) = (\gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \gamma_{i+1} - 1, \gamma_i + 1, \gamma_{i+2}, \cdots, \gamma_n)$$

I like to think of this in the following way: Suppose we have a copies of i, and b copies of $i+1$. Put one $i+1$ aside, switch the other $(i+1)$'s to i's and i's to $(i+1)$'s. For example, $(10, 3)$ becomes $(2, 11)$.

Example: This is a diagram of the SSYT of $\lambda = (21)$. The SSYT related by the involution have arrows between them along with the permutation $s_{i,i+1}^*$ relating their contents. We will describe this involution next time.